

## ANOTHER GENERALIZATION OF THE FIBONACCI AND LUCAS NUMBERS

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**Abstract:** This paper considers some generalizations of the Fibonacci and Lucas numbers which are essentially ratios of the former, and hence not necessarily integers. Nevertheless, some new and elegant results emerge as well as variations on well-established identities.

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### 1. Introduction

Sparked by ideas in [1 and 3], we consider some properties of second-order generalizations of the Fibonacci and Lucas numbers defined by

$$f_n^{(k+1)} = \frac{\alpha^{nk+k} - \beta^{nk+k}}{\alpha^k - \beta^k}. \quad (1.1)$$

and

$$g_n^{(k+1)} = \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha^k - \beta^k} \quad (1.2)$$

where  $\alpha, \beta$  are the roots of

$$x^2 - px + q = 0. \quad (1.3)$$

Clearly,

$$g_{kn}^{(k+1)} = f_n^{(k+1)}$$

Thus, when  $k=1$ ,

$$\begin{aligned} f_n^{(2)} &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= g_n^{(2)} \\ &= u_n \end{aligned}$$

are the Lucas fundamental numbers [7] and Horadam's well-known generalization of the Fibonacci numbers [5]. Trivially then

$$f_n^{(k+1)} u_k = u_{nk+k} \quad (1.4)$$

and

$$g_n^{(k+1)} u_k = u_{n+k} \quad (1.5)$$

but there are also some less obvious properties. Some numerical examples are tabulated later in Section 4 of this paper.

## 2. Recurrence Relations

The elements of the sequence  $\{f_n^{(k+1)}\}$  satisfy the second order recurrence relation

$$f_{n+1}^{(k+1)} = v_k f_n^{(k+1)} - q^k f_{n-1}^{(k+1)} \quad (2.1)$$

in which  $v_k$  is the ordinary generalized Lucas primordial sequence [7].

Proof:

$$\begin{aligned} v_k f_n^{(k+1)} - q^k f_{n-1}^{(k+1)} &= \frac{(\alpha^k + \beta^k)(\alpha^{nk+k} - \beta^{nk+k}) - (\alpha\beta)^k(\alpha^{kn} - \beta^{kn})}{\alpha^k - \beta^k} \\ &= \frac{(\alpha^{nk+2k} - \beta^{nk+2k}) + (\alpha\beta)^k(\alpha^{kn} - \beta^{kn}) - (\alpha\beta)^k(\alpha^{kn} - \beta^{kn})}{\alpha^k - \beta^k} \\ &= \frac{(\alpha^{nk+2k} - \beta^{nk+2k})}{\alpha^k - \beta^k} \\ &= f_{n+1}^{(k+1)} \end{aligned}$$

as required. Similarly, it may be shown that  $g_{n-1}^{(k+1)}$  satisfies the recurrence relation which has (1.3) as an auxiliary equation, namely,

$$g_{n+1}^{(k+1)} = p g_n^{(k+1)} - q g_{n-1}^{(k+1)} \quad (2.2)$$

It then follows that the ordinary generating functions are given (formally) by

$$\sum_{n=0}^{\infty} f_{n+1}^{(k+1)} x^n = \frac{1}{1 - v_k x + q^k x^2} \quad (2.3)$$

and

$$\sum_{n=0}^{\infty} g_{n+1}^{(k+1)} x^n = \frac{1 + (u_k - v_k)x}{1 - px + qx^2} \quad (2.4)$$

Proof of (2.3):

Let

$$f(x) = \sum_{n=0}^{\infty} f_{n+1}^{(k+1)} x^n$$

so that from (1.2)

$$\begin{aligned} (1 - v_k x + q^k x^2) f(x) &= f_0^{(k+1)} + (f_1^{(k+1)} - f_0^{(k+1)} v_k) x \\ &= 1 + \left( \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} - (\alpha^k + \beta^k) \right) x \\ &= 1. \end{aligned}$$

Proof of (2.4):

Let

$$g(x) = \sum_{n=0}^{\infty} g_{n+1}^{(k+1)} x^n$$

so that from (1.2)

$$\begin{aligned} (1 - px + qx^2) g(x) &= g_0^{(k+1)} + (g_1^{(k+1)} - g_0^{(k+1)} p) x \\ &= 1 + \left( \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} - (\alpha^k - \beta^k) \right) x \\ &= 1 + (u_k - v_k) x. \end{aligned}$$

### 3. Lucas Primordial Sequence Connections

As an analogue of Simson's identity we have, not surprisingly, that

$$\left( g_n^{(k+1)} \right)^2 - g_{n-k}^{(k+1)} g_{n+k}^{(k+1)} = q^n. \quad (3.1)$$

Proof:

The numerator of the left hand side reduces to

$$\begin{aligned} (\alpha\beta)^n \alpha^{2k} + (\alpha\beta)^n \beta^{2k} - 2(\alpha\beta)^n (\alpha\beta)^k &= (\alpha\beta)^n (\alpha^k - \beta^k)^2 \\ &= q^n (\alpha^k - \beta^k)^2 \end{aligned}$$

which is  $q^n$  times the denominator of the left hand side, as required.

When  $p = -q = P$  say, we are able to relate the  $f_n^{(k+1)}$  to the ordinary Lucas fundamental numbers  $u_n$  by means of a generalization of a result of Barakat [2] for the ordinary Lucas numbers. Barakat proved that

$$\begin{aligned} u_n &= \sum_{0 \leq m \leq n} \binom{n-m}{m} P^{n-2m} (-q)^m \\ &= \sum_{0 \leq m \leq n} \binom{n-m}{m} P^{n-m}. \end{aligned}$$

We now define

$$\begin{aligned} x_n &= \frac{u_n}{P} \\ &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^2 - \beta^2} \end{aligned}$$

and set  $y_n = x_{n+1}$  for later notational convenience, so that from Simson's identity

$$u_k u_k - u_{k-1} u_{k+1} = (-P)^k$$

we have

$$x_k y_{k-1} - x_{k-1} y_k = (-P)^{k-2}. \quad (3.2)$$

From this we can establish a connection with the Lucas primordial numbers, namely

$$P y_{k-1} + P^2 y_{k-3} = v_k. \quad (3.3)$$

We are now in a position to assert a property which relates these generalized Fibonacci numbers to the ordinary Fibonacci numbers, and, at the same time, yields an iterative formula for the general term [9]. This formula generalizes [2] and [8] and uses a result from [4].

Theorem:

$$f_n^{(k+1)} = \sum_{0 \leq m+s \leq n} \binom{m}{s} \binom{n-m}{s} u_{k-2}^{m-s} u_{k-1}^{2s} u_k^{n-m-s} P^m. \quad (3.4)$$

Proof:

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(k+1)} z^n &= (1 - v_k z + (-P)^k z^2)^{-1} && \text{(from (2.3))} \\ &= (1 - (P^2 y_{k-3} + P y_{k-1}) z + (-P)^k z^2)^{-1} && \text{(from (3.3))} \\ &= (1 - (P^2 x_{k-2} + P y_{k-1}) z + P^3 (x_{k-2} y_{k-1} - x_{k-1} y_{k-2}) z^2)^{-1} && \text{(from (3.2))} \\ &= ((1 - P^2 x_{k-2} z)(1 - P y_{k-1} z) - P^3 x_{k-1} y_{k-2} z^2)^{-1} \\ &= \sum_{s=0}^{\infty} (1 - P^2 x_{k-2} z)^{-s-1} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s y_{k-2}^s P^{3s} z^{2s} \\ &= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+s}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^m y_{k-2}^s P^{3s+2m} z^{2s+m} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^m \binom{m}{s} (1 - P y_{k-1} z)^{-s-1} x_{k-1}^s x_{k-2}^{m-s} y_{k-2}^s P^{s+2m} z^{m+s} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^m \sum_{t=0}^{\infty} \binom{m}{s} \binom{t+s}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^t y_{k-2}^s P^{s+2m+t} z^{m+s+t} \\ &= \sum_{n=0}^{\infty} \sum_{0 \leq m+s \leq n} \binom{m}{s} \binom{n-m}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^{n-m-s} y_{k-2}^s P^{n+m} z^n. \end{aligned}$$

So, on equating coefficients of  $z^n$  we find that

$$\begin{aligned} f_n^{(k+1)} &= \sum_{0 \leq m+s \leq n} \binom{m}{s} \binom{n-m}{s} \frac{u_{k-1}^{(2)s} u_{k-2}^{(2)m-s} u_k^{(2)n-m-s} u_{k-1}^{(2)s}}{P^{s+m-s+n-m-s+s}} P^{n+m} \\ &= \sum_{0 \leq m+s \leq n} \binom{m}{s} \binom{n-m}{s} (u_{k-2}^{(2)})^{m-s} (u_{k-1}^{(2)})^{2s} (u_k^{(2)})^{n-m-s} P^m \end{aligned}$$

as required.

For example, when  $k=1$ , since  $f_{-1}^{(2)} = 0$ ,

$$\begin{aligned} f_n^{(2)} &= \sum_{0 \leq 2m \leq n} \binom{n-m}{s} P^{n-2m} P^m \\ &= \sum_{0 \leq 2m \leq n} \binom{n-m}{s} P^{n-m} \end{aligned}$$

which agrees with the result due to Barakat above.

#### 4. Concluding Comments

Other properties can be readily developed to relate these generalizations to other parts of Fibonacci and Lucas theory. For instance, we can prove that

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\} = \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)} \quad (4.1)$$

in which  $\left\{ \begin{matrix} n \\ j \end{matrix} \right\}$ , a Fibonomial coefficient [6], is an analogue of the binomial coefficient, defined in the context of this paper by:

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\} = \frac{u_n^{(2)} u_{n-1}^{(2)} \dots u_{n-j+1}^{(2)}}{u_0^{(2)} u_1^{(2)} \dots u_{j-1}^{(2)}}.$$

Proof of (4.1):

$$\begin{aligned} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} &= \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^n - \beta^n) \dots (\alpha^{n-j+2} - \beta^{n-j+2})}{(\alpha - \beta)(\alpha^2 - \beta^2) \dots (\alpha^j - \beta^j)} \\ &= g_n^{(2)} g_{n-2}^{(3)} \dots g_{n-2j+2}^{(j+1)} \\ &= \prod_{i=2}^{j+1} g_{n-2i+4}^{(i)}, \end{aligned}$$

as required.

The  $n$ th Fermatian of index  $x$  is defined formally by [Whitney]

$$\underline{x}_n = 1 + x + x^2 + \dots + x^{n-1}.$$

Then

$$f_n^{(k+1)} = \beta^{kn} \left( \left( \frac{\alpha}{\beta} \right)^k \right)_{n+1} . \quad (4.2)$$

Proof:

$$\begin{aligned} f_n^{(k+1)} &= \frac{(\alpha^k)^{n+1} - (\beta^k)^{n+1}}{(\alpha^k) - (\beta^k)} \\ &= \beta^{kn} \left( 1 + \left( \frac{\alpha}{\beta} \right) + \dots + \left( \frac{\alpha}{\beta} \right)^n \right) \\ &= \beta^{kn} \left( \left( \frac{\alpha}{\beta} \right)^k \right)_{n+1} . \end{aligned}$$

Other arbitrary order generalizations of the Fibonacci and Lucas numbers have been produced by various alterations to the characteristic equations [4]. We conclude with the following table of the first seven values of some of the sequences discussed in this paper.

$n$	0	1	2	3	4	5	6	...
$f_n^{(2)}$	1	1	2	3	5	8	13	...
$f_n^{(3)}$	1	3	8	21	55	144	377	...
$f_n^{(4)}$	1	3	13	55	233	987	4181	...
$g_n^{(2)}$	1	1	2	3	5	8	13	...
$g_n^{(3)}$	1	2	3	5	8	13	21	...
$g_n^{(4)}$	1	1½	2½	4	6½	10½	17	...

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