NNTDM 16 (2010), 2, 29-32

Rows of Odd Powers in the Modular Ring Z_4

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Abstract

Simple functions were obtained for the rows of odd powers in the modular ring Z_4, wherein integer N is represented by $N = 4r_i + i$, $i = 0,1,2,3$. These row functions are based on the row functions for squares. When $3 \nmid N$, the row of $N^2 = 3n(3n \pm 1)$, or when 3|*N*, the row of $N^2 = 2 + 18(\frac{1}{2}n(n+1)), n = 0,1,2,3,...$.

Keywords: Primitive Pythagorean triples, modular rings, triangular numbers, pentagonal numbers

AMS Classification Numbers: 11A41, 11A07

1. Introduction

Integer structure analysis with modular rings provides the tools for new perspectives on well-known quantities and functions, such as infinite series, integer powers, primitive Pythagorean triples (pPts), and so on [4]. A pPt is a Pythagorean triple (*x*,*y*,*z*) such that, the greatest common divisor of (*x*,*y*,*z*) is unity [1]. For instance, a pPt such as

$$625=576 + 49 \tag{1.1}$$

might not seem at first glance to have any common factors. However, $625 \in \bar{1}_4 \subset Z_4$ (Table 1). That is, 625 = 4 x 156 + 1. On the other hand, $576 \in \bar{0}_4$, and so 576 = 4 x 144, whereas $49 \in \bar{1}_4$, with 49 = 4 x 12 + 1. Thus, Equation (1.1) becomes

$$13 = 12 + 1. \tag{1.2}$$

Different relationships among pPts may be explored on this basis, including Lehmer's intriguing result that.the fraction of primitive triples $N(p)$ with perimeter less than p is [3]

$$\lim_{p\to\infty}\frac{N(p)}{p}=\frac{\ln 2}{\pi.}=0.070230..$$

In the context of this paper we are more concerned with the functions of the rows of squares of integers, N. When $3|N$, the rows follow the triangular numbers, but when $3\nmid N$, the rows follow the pentagonal numbers [2,4]. The functions for the rows of squares can then be used to analyse primes, p, by means of

$$p=4r_1+1=x^2+y^2$$

to develop a general equation for these primes in modular rings [5]. In this paper we extend the functions to powers greater than 2.

Row	$F(r)$	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23

Table 1: Modular Ring Z_4

Class of *N*	**3|*N***	**row of *N***	***n***	**row of N^2**
$\bar{1}_4$	No	$\frac{3}{2}n$	even	$3n(3n+1)$
		$\frac{1}{2}(3n-1)$	odd	$3n(3n-1)$
$\bar{3}_4$	No	$\frac{1}{2}(3n-2)$	even	$3n(3n-1)$
		$\frac{1}{2}(3n-1)$	odd	$3n(3n+1)$
$\bar{1}_4$	Yes	$\frac{1}{2}(3n+1)$	odd only	$2+9n(n+1)$
$\bar{3}_4$	Yes	$\frac{3}{2}n$	even only	$2+9n(n+1)$

Table 2: rows of N^2

2. Rows of Cubes in Z_4

Classes $\bar{1}_4$ and $\bar{3}_4$ contain the odd integers (Table 1). When $N=4r_1+1$, $N^3\in\bar{1}_4$ too, and for $N=4r_3+3$ and $N^3\in\bar{3}_4$. This is in contrast to the squares which are all elements of Class $\bar{1}_4$. Since

$$N^3=N^2N \tag{2.1}$$

we may use the row functions of the squares. For instance, consider $N\in\bar{1}_4$ and $3\nmid N$. Then with

$$N=4(6K)+1,\ K=\tfrac{1}{2}n(3n\pm 1) \tag{2.2}$$

and

$$N = 4r_1 + 1, \tag{2.3}$$

then

$$N^3 = (24K+1)(4r_1+1) \tag{2.4}$$

so R_1, the row of N^3, is given by

$$R_1 = 24Kr_1 + 6K + r_1. \tag{2.5}$$

When n is even,

$$r_1 = \tfrac{3}{2}n,$$

when n is odd,

$$r_1 = \tfrac{1}{2}(3n-1).$$

These functions are obtained by comparing the square functions with the rows of N (Table 2). Substituting in the $f(n)$ for K and r_1 in Table 2 yields the results in Table 3.

Class of *N*	**Row of *N***	***n***	***K***	**Row of cube**
$\bar{1}_4, 3 \nmid N$	$\frac{3}{2}n$	even	$\frac{1}{2}(3n+1)$	$3n(3n+1)(6n+1)+\frac{3}{2}n$
	$\frac{1}{2}(3n-1)$	odd	$\frac{1}{2}(3n-1)$	$3n(3n-1)(6n-1)+\frac{1}{2}(3n-1)$
$\bar{3}_4, 3 \nmid N$	$\frac{1}{2}(3n-2)$	even	$\frac{1}{2}(3n-1)$	$3n(3n-1)(6n-1)+\frac{1}{2}(3n-2)$
	$\frac{1}{2}(3n-1)$	odd	$\frac{1}{2}(3n+1)$	$3n(3n+1)(6n+1)+\frac{1}{2}(3n-1)$
$\bar{1}_4, 3 \mid N$	$\frac{1}{2}(3n+1)$	odd only	-	$3(2+9n(n+1))(2n+1)+\frac{1}{2}(3n+1)$
$\bar{3}_4, 3 \mid N$	$\frac{3}{2}n$	even only	-	$3(2+9n(n+1))(2n+1)+\frac{3}{2}n$

Table 3: N^3 data

Since $\bar{3}_4 \times \bar{3}_4 \in \bar{1}_4$, when $N \in \bar{3}_4$,

$$N^3 = (24K+1)(4r_3+3) \tag{2.6}$$

which gives the $f(n)$ in Table 3.

For $3|N$, the functions in Table 2 are substituted for $6K$ and r_1 and r_3 in Equation (2.5) and the row of N in Equation (2.6) to give the functions in Table 3.

3.Rows of N^5 in Z_4

Obviously all odd powers may be split into $N^2N^2N^2 \ldots N$. For example,

$$N^5 = (N^2)^2 N \tag{3.1}$$

$$= (24K+1)^2(4r_1+1)$$

$$= (24K+1)^2(4r_3+3) \tag{3.2}$$

Substitution of the values of the various quantities from Table 2 gives the $f(n)$ listed in Table 4.

4. Even Integers

Since all even integers may be put in the form

$$M = 2^t q \tag{4.1}$$

where q is odd ($(4r_1+1)$ or $(4r_3+3)$, $r_i = 0,1,2,3,\ldots$).

Thus,

$$M^s = 2^{ts} q^s$$

so that rows may be found using the above analysis.

Class of *N*	**Row of *N***	***n***	***K***	**Row of N^5**
$\bar{1}_4, 3 \nmid N$	$\frac{3}{2}n$	even	$\frac{1}{2}(3n+1)$	$6n(3n+1)(6n+1)(6n(3n+1)+1)+\frac{3}{2}n$
	$\frac{1}{2}(3n-1)$	odd	$\frac{1}{2}(3n-1)$	$6n(3n-1)(6n-1)(6n(3n-1)+1)+\frac{1}{2}(3n-1)$
$\bar{3}_4, 3 \nmid N$	$\frac{1}{2}(3n-2)$	even	$\frac{1}{2}(3n-1)$	$6n(3n-1)(6n-1)(6n(3n-1)+1)+\frac{1}{2}(3n-2)$
	$\frac{1}{2}(3n-1)$	odd	$\frac{1}{2}(3n+1)$	$6n(3n+1)(6n+1)(6n(3n+1)+1)+\frac{1}{2}(3n-1)$
$\bar{1}_4, 3 \mid N$	$\frac{1}{2}(3n+1)$	odd only	-	$6(2+9n(n+1))(2n+1)(18n(n+1)+5)+\frac{1}{2}(3n+1)$
$\bar{3}_4, 3 \mid N$	$\frac{3}{2}n$	even only	-	$6(2+9n(n+1))(2n+1)(18n(n+1)+5)+\frac{3}{2}n$

Table 4: N^5 data

5. Final Comments

When analyzing higher powered analogs of Pythagorean triples it should be useful to break down the components into the classes and have functions for the rows so that the powers are first reduced to rather simple functions of *n*, as in [6].

The foregoing analysis may be readily extended to even powers since

$$N^{2m} = N^2 N^2 \ldots . \tag{5.1}$$

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