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Combined 2-Fibonacci sequences

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The Fibonacci sequence 0,1,1,2,3,5,... is an object of different modifications and extensions.

In [7, 2, 16] four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described and called *2-Fibonacci sequences* (or *2-F-sequences).* The four schemes are the following

$$\begin{array}{ll} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d & \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0 & \alpha_{n+2} = \alpha_{n+1} + \beta_n,\ n \geq 0 \\ \beta_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0 & \beta_{n+2} = \beta_{n+1} + \alpha_n,\ n \geq 0 \end{array}$$

$$\begin{array}{ll} \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d & \alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ \alpha_{n+2} = \beta_{n+1} + \alpha_n,\ n \geq 0 & \alpha_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0 \\ \beta_{n+2} = \alpha_{n+1} + \beta_n,\ n \geq 0 & \beta_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0 \end{array}$$

Obviously, the Third and the Fourth schemes contain two standard Fibonacci sequences and therefore they are trivial modification, while the first two schemes are essential extensions of Fibonacci sequence.

Clearly, if we set $a = b$ and $c = d$, then sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is called a generalized Fibonacci sequence, where

$$F_0(a, c) = a,$$

$$F_1(a, c) = c,$$

$$F_{n+2}(a, c) = F_{n+1}(a, c) + F_n(a, c).$$

Let $F_i = F_i(0, 1)$; $\{F_i\}_{i=0}^{\infty}$ be the ordinary Fibonacci sequence.

Once introduced, the idea for 2-F-sequences became an object of subsequent research (see, e.g., [1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22]). These sequences were extended to 3-Fibonacci sequences and to 2-Tribonacci sequences. The possibility for defining k-Fibonacci sequences, m-Tribonacci sequences, and more generally – k-m-bonacci sequences are mentioned.

Here, we will introduce two new schemes of a Fibonacci sequence, that are combinations of the first two 2-F-sequences, mentioned above. The first of them has the form:

$$\alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d$$
$$\alpha_{n+2} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n,\ n \geq 0$$
$$\beta_{n+2} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n,\ n \geq 0$$

The first 10 members of the new scheme have the form shown on Table 1.

Table 1

α_n	β_n
$2a$	$2b$
$2c$	$2d$
$2b+c+d$	$2a+c+d$
$a+b+c+3d$	$a+b+3c+d$
$3a+b+3c+3d$	$a+3b+3c+3d$
$3a+3b+6c+4d$	$3a+3b+4c+6d$
$4a+6b+8c+8d$	$6a+4b+8c+8d$
$8a+8b+12c+14d$	$8a+8b+14c+12d$
$14a+12b+21c+21d$	$12a+14b+21c+21d$
$21a+21b+35c+33d$	$21a+21b+33c+35d$
$\dots$	$\dots$

where a, b, c, d are given constants.

Let σ be the integer function defined for every $k \geq 0$ by:

r	$\sigma(4.k+r)$
0	0
1	1
2	0
3	-1

Obviously, for every $n \geq 0$,

$$\sigma(n+2) + \sigma(n) = 0.$$

THEOREM 1. For every natural number $n \geq 0$

$$\alpha_{n+2} = (F_{n+1} + \sigma(n-1)).a + (F_{n+1} + \sigma(n+1)).b + (F_{n+2} + \sigma(n+2)).c + (F_{n+2} + \sigma(n)).d$$

$$\beta_{n+2} = (F_{n+1} + \sigma(n+1)).a + (F_{n+1} + \sigma(n-1)).b + (F_{n+2} + \sigma(n)).c + (F_{n+2} + \sigma(n+2)).d$$

The proof of this assertion can be made, for example, by induction.

For $n = 0$ we see the validity of the two formulas from Table 1. Let us assume that these formulas are valid for some natural number $n \geq 0$. Then

$$\alpha_{n+3} = \frac{\alpha_{n+2} + \beta_{n+2}}{2} + \beta_{n+1}$$

$$= \frac{1}{2}.((F_{n+1} + \sigma(n-1)).a + (F_{n+1} + \sigma(n+1)).b + (F_{n+2} + \sigma(n+2)).c + (F_{n+2} + \sigma(n)).d)$$

$$+(F_{n+1} + \sigma(n+1)).a + (F_{n+1} + \sigma(n-1)).b + (F_{n+2} + \sigma(n)).c + (F_{n+2} + \sigma(n+2)).d)$$

$$+(F_n + \sigma(n)).a + (F_n + \sigma(n-2)).b + (F_{n+1} + \sigma(n-1)).c + (F_{n+1} + \sigma(n+1)).d$$

$$= F_{n+1}.a + F_{n+1}.b + F_{n+2}.c + F_{n+2}.d$$

$$+(F_n + \sigma(n)).a + (F_n + \sigma(n-2)).b + (F_{n+1} + \sigma(n-1)).c + (F_{n+1} + \sigma(n+1)).d$$

$$= (F_{n+2} + \sigma(n)).a + (F_{n+2} + \sigma(n-2)).b + (F_{n+3} + \sigma(n-1)).c + (F_{n+3} + \sigma(n+1)).d$$

$$= (F_{n+2} + \sigma(n)).a + (F_{n+2} + \sigma(n+2)).b + (F_{n+3} + \sigma(n-1)).c + (F_{n+3} + \sigma(n+1)).d$$

The formula for β_{n+3} may be checked in similar manner.

The second new sequence has the form:

$$\alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d$$

$$\alpha_{n+2} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \alpha_n,\ n \geq 0$$

$$\beta_{n+2} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \beta_n,\ n \geq 0$$

The first 10 members of the new scheme have the form shown on Table 2.

Table 2

α_n	β_n
$2a$	$2b$
$2c$	$2d$
$2a + c + d$	$2b + c + d$
$a + b + 3c + d$	$a + b + c + 3d$
$3a + b + 3c + 3d$	$a + 3b + 3c + 3d$
$3a + 3b + 6c + 4d$	$3a + 3b + 4c + 6d$
$6a + 4b + 8c + 8d$	$4a + 6b + 8c + 8d$
$8a + 8b + 14c + 12d$	$8a + 8b + 12c + 14d$
$14a + 12b + 21c + 21d$	$12a + 14b + 21c + 21d$
$21a + 21b + 35c + 33d$	$21a + 21b + 33c + 35d$
...	...

where a, b, c, d are given constants.

Let ρ be the integer function defined for every $k \geq 0$ by:

r	$\rho(2.k + r)$
0	1
1	0

THEOREM 2. For each natural number $n \geq 0$

$$\alpha_{n+2} = (F_{n+1} + \rho(n)).a + (F_{n+1} - \rho(n)).b + (F_{n+2} + \rho(n+1)).c + (F_{n+2} - \rho(n+1)).d$$

$$\beta_{n+2} = (F_{n+1} - \rho(n)).a + (F_{n+1} + \rho(n)).b + (F_{n+2} - \rho(n+1)).c + (F_{n+2} + \rho(n+1)).d$$

The proof of this assertion is similar to the above one.

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