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A remark on an arithmetic function. Part 3

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1 Introduction

In a series of papers the author studied some properties of the well-known arithmetic functions φ and σ (see, e.g. [5, 7]). With respect to these research, a new function was defined in [1, 2] and some of its properties were discussed there. Here we shall continue this research.

Firstly, following [1, 2], for the natural number $n \geq 2$:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ... $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers, we shall define the following functions:

$$\zeta(n) = \sum_{i=1}^{k} \alpha_i . p_i, \ \zeta(1) = 1,$$

$$\underline{\text{set}}(n) = \{p_1, p_2, ..., p_k\}, \ \underline{\text{set}}(1) = \{1\},$$

$$\underline{\text{cas}}(n) = k, \ \underline{\text{cas}}(1) = 1,$$

$$\dim(n) = \sum_{i=1}^{k} \alpha_i, \ \dim(1) = 1.$$

2 New properties of function ζ

Theorem 1: For every natural number n

$$\zeta(n)\underline{\text{cas}}(n) \leq n + 4 \tag{1}$$

Proof Let n be a prime number. Then, from (1) we obtain:

$$n + 4 - \zeta(n)\underline{\text{cas}}(n) = n + 4 - n.1 = 4 \geq 0.$$

Let us assume that (1) is valid for an arbitrary natural number n with $\dim(n) = k$ and let p be an arbitrary prime number. For p there are two cases.

Case 1: $p \in \underline{\text{set}}(n)$. Therefore, $\underline{\text{cas}}(np) = \underline{\text{cas}}(n)$ and

$$np + 4 - \zeta(np)\underline{\text{cas}}(np) = n.p + 4 - (\zeta(n) + p).\underline{\text{cas}}(n)$$

$$= n.(p-1) + n + 4 - \zeta(n).\underline{\text{cas}}(n) - p.\underline{\text{cas}}(n)$$

(from (1))

$$\geq n.(p-1) - p.\underline{\text{cas}}(n)$$

$$= \frac{n}{2}.2.(p-1) - p.\underline{\text{cas}}(n) \geq 0,$$

because, obviously, for every natural number n: $n \geq 2\underline{\text{cas}}(n)$ and for every prime number p: $2(p-1) \geq p$.

Case 2: $p \notin \underline{\text{set}}(n)$. Therefore, $\underline{\text{cas}}(np) = \underline{\text{cas}}(n) + 1$. If n is a prime number, then

$$np + 4 - \zeta(np)\underline{\text{cas}}(np) = n.p + 4 - 2.(n + p) \geq 0$$

and we obtain "=" only for $n = 2$.

Let $\dim(n) \geq 2$. Then,

$$np + 4 - \zeta(np)\underline{\text{cas}}(np) = n.p + 4 - (\zeta(n) + p).(\underline{\text{cas}}(n) + 1).$$

If $p = 2$, then by the condition that $p \notin \underline{\text{set}}(n)$, it follows that n is an odd number. Therefore,

$$np + 4 - \zeta(np)\underline{\text{cas}}(np) = 2n + 4 - (\zeta(n) + 2).(\underline{\text{cas}}(n) + 1)$$

$$= 2n + 4 - \zeta(n)\underline{\text{cas}}(n) - \zeta(n) - 2.\underline{\text{cas}}(n) - 2$$

(from (1))

$$\geq n - \zeta(n) - 2.\underline{\text{cas}}(n) - 2$$

(for each natural number $n \geq 31$ we can check that: $\zeta(n) \geq 2.\underline{\text{cas}}(n) + 6$)

$$\geq n - 2.\zeta(n) + 4 \geq 0$$

(by assumption, $n = q.m$, where $q \geq 3$ is a prime number and by inductive assumption $m + 4 - \zeta(m)\underline{\text{cas}}(m) \geq 0$)

$$= q.m - 2.\zeta(q.m) + 4 = q.m - 2.(q + m) + 4$$

$$= q.(m-2) - 2.m + 4 \geq 3.(m-2) - 2.m + 4 = m - 2 \geq 0.$$

Finally, let $p > 2$. Then,

$$np + 4 - \zeta(np)\underline{\text{cas}}(np) = n(p-1) + n + 4 - (\zeta(n) + p).(\underline{\text{cas}}(n) + 1)$$

$$= n(p-1) + n + 4 - \zeta(n)\underline{\text{cas}}(n) - \zeta(n) - p.\underline{\text{cas}}(n) - p$$

(from (1))

$$\geq n(p-1) - \zeta(n) - p.\underline{\text{cas}}(n) - p$$

(we can directly check that $\zeta(n) \geq \underline{\text{cas}}(n) + 1$)

$$\geq n(p-1) - (p+1).\zeta(n) \geq 0,$$

because, obviously, for every natural number n: $n \geq 2.\zeta(n)$ and for every prime number $p \geq 3$: $2(p-1) \geq p+1$.

Therefore, for $n \geq 31$ the assertion is valid. By a direct check we see that

n	$n+4-\zeta(n)\underline{\text{cas}}(n)$	n	$n+4-\zeta(n)\underline{\text{cas}}(n)$
1	4	16	4
2	4	17	4
3	4	18	6
4	4	19	4
5	4	20	6
6	0	21	5
7	4	22	0
8	6	23	4
9	7	24	10
10	0	25	19
11	4	26	0
12	2	27	22
13	4	28	10
14	0	29	4
15	3	30	4

Hence, Theorem 1 is valid for every natural number n.

If we like to construct an inequality with the same components, but in the opposite direction, it could have the form

$$\zeta(n)^{\underline{\text{cas}}(n)} \geq n,$$

but there are counterexamples, e.g.

$$\zeta(1000)^{\underline{\text{cas}}(1000)} = (3.2 + 3.5)^2 = 21^2 < 1000.$$

One of the possible inequalities is discussed in the following

Theorem 2: For every natural number n

$$\zeta(n)^{\dim(n)} \geq n. \tag{2}$$

Proof Let n be a prime number. Then, from (2) we obtain:

$$\zeta(n)^{\dim(n)} - n = n^1 - n = 0.$$

Let us assume that (2) is valid for an arbitrary natural number n with $\dim(n) = k$, and let p be an arbitrary prime number. Then,

$$\zeta(n.p)^{\dim(n.p)} - n.p = (\zeta(n) + p)^{\dim(n)+1} - n.p$$

(from (2))

$$\geq (\zeta(n)+p)^{\dim(n)+1} - \zeta(n)^{\dim(n)}.p$$

$$= \zeta(n)^{\dim(n)}((\zeta(n)+p).(1+\frac{p}{\zeta(n)})^{\dim(n)} - p)$$

$$> \zeta(n)^{\dim(n)}(\zeta(n)+p-p) > 0.$$

Therefore, (2) is valid.

3 Function ζ and n-th prime number

Following idea from [3, 4], where we introduced four new formulae for the well-known function $\pi(n)$ (see, e.g. [5, 7]) and a new formula for the n-th prime number p_n, now we shall introduce another (simpler) formula for $\pi(n)$ and p_n.

Let us define functions sg and $\overline{sg}$ by:

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

$$\overline{sg}(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases},$$

where x is a real number.

THEOREM 3: The following equality holds for every natural number $n \geq 2$:

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(k - \zeta(k)).$$

Proof: For every natural number k, such that $k \leq n$, if k is prime, then $k = \zeta(k)$ and hence $\overline{sg}(k - \zeta(k)) = 1$. On the other hand, if k is not prime, then $k - \eta(k) > 0$, i.e., $\overline{sg}(k - \eta(k)) = 0$. Therefore, the sum is equal to $\pi(n)$.

Of course, $\pi(0) = 0$ and $\pi(1) = 0$.

For the so constructed formula for $\pi(n)$, we can proveby analogy with [3, 4] the following

THEOREM 4: For every natural number n:

$$p_n = \sum_{i=0}^{C(n)} sg(n - \pi(i)),$$

where (see [6])

$$C(n) = [\frac{n^2+3n+4}{4}].$$

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