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On the linear systems over rhotrices

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Abstract

Let A, B and C be rhotrices. In this paper we investigate systems of the form $AB = C$ and come up with conditions necessary for their solvability. We also outline a direct procedure for computing an nth root of a rhotrix.

Keywords: Rhotrix; Linear system; Heart of a rhotrix

1. Introduction

Mathematical arrays that are in some way between two-dimensional vectors and 2×2 dimensional matrices were suggested by Atanassov and Shannon [4]. As an extension to this idea, Ajibade [1] introduced an object that lies between 2×2 dimensional matrices and 3×3 dimensional matrices called 'rhotrix'. A rhotrix as given in [1] is of the form

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle, \tag{1}$$

where $a, b, d, e, h(R) \in \mathbb{R}$, and $h(R)$ is called the heart of a rhotrix *R*. A rhotrix of the form (1) is called based rhotrix, which is rhotrix of base three. It was also mentioned in [1] that a rhotrix can be extended to n-dimension. If the n-dimensional rhotrix is denote by R_n and $|R_n|$ the number of elements of R_n, then $|R_n| = \frac{1}{2}(n^2+1)$.

The initial algebra and analysis of rhotrices was also presented in [1]. Let *R* and *Q* be two rhotrices such that

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{ and } Q = \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle. \tag{2}$$

The addition and multiplication of rhotrices R and Q defined by Ajibade [1] are as follows:

$$R + Q = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle,$$

$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle .$$

In an attempt to answer some questions raised by Ajibade [1], Sani [6] developed another multiplication method for rhotrices called *row-column multiplication.* The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the matrices R and Q defined in (2) as follows:

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle .$$

One of the advantages of row-column multiplication method is that the row and column of rhotrices are spelt out and can easily be identified. For instance a 5-dimensional rhotrix is the following

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{12} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle ,$$

where a_{ij}, c_{lk} for $i,j=1,2,3$ and $l,k=1,2$ is the element on the ith row and jth column.

A generalization of the row-column multiplication method for n-dimensional rhotrices was given by Sani [7]. That is: given n-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ the multiplication of R_n and Q_n is as follows

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \; t=(n+1)/2 .$$

The method of converting a rhotrix to a special matrix called 'coupled matrix' was suggested by Sani [8]. This idea was used to solve systems of $n \times n$ and $(n-1)\times(n-1)$ matrix problems simultaneously. The concept of vectors, one-sided system of equations and eigenvector eigenvalue problem in rhotrices were introduced by Aminu [2]. A necessary and sufficient condition for the solvability of one sided system of rhotrix was also presented in [2]. If a system is solvable it was shown how a solution can be found. Rhotrix vector spaces and their properties were presented by Aminu [3].

In this paper we investigate linear systems of the form $AB = C$ where A, B, C are rhotrices and the multiplication method defined in [1] is applied.

2. Rhotrix and its basic properties

Let $t=(n+1)/2$ for $n \in \square$. By 'rhotrix' we understand an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices. The diagonal rhotrix will be denoted by I and is given by $I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$, if the multiplication defined in [1] is used, and $I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$ if the row column multiplication method is used.

We also denote by 0 the usual zero, which is the neutral element under addition and for convenience we use the same symbol to denote any rhotrix or rhotrix vector whose every component is 0.

We will now summarize some basic properties of rhotrices. The following properties hold for any rhotrices A, B and C over □ and $\alpha \in$ □ :

$$A + 0 = 0 + A = A$$
$$A + B = B + A$$
$$(A + B) + C = A + (B + C)$$
$$\alpha(A + B) = \alpha A + \alpha B$$
$$A(B + C) = AB + AC$$
$$A(BC) = (AB)C$$
$$AI = A = IA$$

3. Linear systems

Let *A, B* and *C* be rhotrices over□ , in this section we will investigate the solvability of linear system of the form *AB=C* when the multiplication defined in [1] is used. The reader should note that the linear of the form $R_n x = b$ has been studied in [2], where R_n is an *n*-dimensional rhotrix and b the right hand side rhotrix vector. We shall without of generality assume that A*, B* and *C* are base rhotrices, that is rhotrix of three dimension.

Consider the linear system *AB=C,* where A*, B* and *C* are base rhotrices. This can be written as

$$AB = C = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & b_1 & \\ b_2 & h(B) & b_3 \\ & b_4 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & h(C) & c_3 \\ & c_4 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a_1 h(B) + b_1 h(A) & \\ a_2 h(B) + b_2 h(A) & h(A)h(B) & a_3 h(B) + b_3 h(A) \\ & a_4 h(B) + b_4 h(A) & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & h(C) & c_3 \\ & c_4 & \end{array} \right\rangle$$

This is equivalent to

$$\left. \begin{array}{c} a_1 h(B) + b_1 h(A) = c_1 \\ a_2 h(B) + b_2 h(A) = c_2 \\ h(A)h(B) = h(C) \\ a_3 h(B) + b_3 h(A) = c_3 \\ a_4 h(B) + b_4 h(A) = c_4 \end{array} \right\}. \qquad (3)$$

Proposition 3.1 *Let A, B and C be rhotrices over*□ *. Then the system AB=C has a unique solution if and only if* $h(A) \neq 0$ *and* $h(C) \neq 0$.

Proof. Suppose $h(A) \neq 0$ and $h(C) \neq 0$. It follows from (3) that

$$\begin{aligned} & h(A) \neq 0 \text{ and } h(C) \neq 0 \\ \Leftrightarrow \quad & h(B) = h(C)/h(A) \text{ and} \\ & b_i = \frac{c_i h(A) - a_i h(C)}{h(A)^2}, i = 1,\ldots,4 \end{aligned} \tag{4}$$

and the theorem statement now follows.

It follows from Proposition 3.1 that we can easily determine an exact unique solution to the system *AB*=*C* if the necessary and sufficient condition is satisfied. This condition is $h(A) \neq 0$ *and* $h(C) \neq 0$.

Proposition 3.2 *Let A, B and C be rhotrices over*□ *. The system AB=C has infinite number of solutions if and only if* $h(A) = h(C) = 0$.

Proposition 3.3 *Let A, B and C be rhotrices over*□ *. The system AB=C has no solution if and only if* $h(A) = 0$ *and* $h(C) \neq 0$.

3.1 An example

Consider the linear system of rhotrix AB=*C* in which

$$A = \left\langle \begin{array}{ccc} & 5 & \\ 3 & 4 & 8 \\ & 6 & \end{array} \right\rangle \text{ and } C = \left\langle \begin{array}{ccc} & 0 & \\ -1 & 12 & 6 \\ & 7 & \end{array} \right\rangle.$$

Find the rhotrix *B* such that *AB*=*C.*

We now make use of (4) to find the rhotrix *B*. The heart of *B* is given as

$$h(B) = \frac{h(C)}{h(A)} = \frac{12}{4} = 3.$$

Entries of *B* can be determined as follows

$$b_1 = \frac{c_1 h(A) - a_1 h(C)}{h(A)^2} = -3\,3/4,$$

$$b_2 = \frac{c_2 h(A) - a_2 h(C)}{h(A)^2} = -2\,1/2,$$

$$b_3 = \frac{c_3 h(A) - a_3 h(C)}{h(A)^2} = -4\,1/2 \text{ and}$$

$$b_4 = \frac{c_4 h(A) - a_4 h(C)}{h(A)^2} = -2\,3/4.$$

Hence, $B = \left\langle \begin{array}{ccc} & -3\frac{3}{4} & \\ -2\frac{1}{2} & 3 & -4\frac{1}{2} \\ & -2\frac{3}{4} & \end{array} \right\rangle.$

4. Square root of a rhotrix

Let A be a rhotrix with positive heart, in this section we will devise a procedure for finding a square root of A. This procedure will only require the multiplication method given in [1] and is motivated by a similar question for matrices as follows

Problem 4.1[5] *Given a real symmetric positive definite* 3×3 *matrix A, outline a direct procedure not involving the singular values or eigenvalues of A for computing a real symmetric positive definite* 3×3 *matrix B satisfying* $B^2 = A$.

Since we want to find the square root of a given rhotrix A, it is equivalent to finding another rhotrix B such that $A = B^2$. Now let

$$A = \left\langle \begin{matrix} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{matrix} \right\rangle \text{ and } B = \left\langle \begin{matrix} & b_1 & \\ b_2 & h(B) & b_3 \\ & b_4 & \end{matrix} \right\rangle.$$

Therefore we want to find b_i for $i = 1,\ldots,4$ and $h(B)$ such that

$$\left\langle \begin{matrix} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 2b_1h(B) & \\ 2b_2h(B) & h(B)^2 & 2b_3h(B) \\ & 2b_4h(B) & \end{matrix} \right\rangle. \qquad (4)$$

It follow from (4) that

$$h(B) = \sqrt{h(A)} \text{ and } b_i = \frac{a_i}{2\sqrt{h(A)}}, i =,1,\ldots,4. \qquad (5)$$

4.1 An example

Consider the following rhotrix and find its corresponding square root

$$A = \left\langle \begin{matrix} & 2 & \\ 1 & 9 & 1 \\ & 3 & \end{matrix} \right\rangle.$$

Using (5) the square root of A is

$$B = \left\langle \begin{matrix} & 1/3 & \\ 1/6 & 3 & 1/6 \\ & 1/2 & \end{matrix} \right\rangle.$$

5. nth root of a rhotrix

Let $A = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{array} \right\rangle$ be a rhotrix with positive heart. The nth power of A is the following

$$A^n = \left\langle \begin{array}{ccc} & na_1h(A) & \\ n\,a_2h(A) & h(A)^n & na_3h(A) \\ & na_4h(A) & \end{array} \right\rangle$$

Since we can easily determine the nth power of a rhotrix, we can use it evaluate the nth root of a given positive rhotrix. It can easily be verified that $B = \left\langle \begin{array}{ccc} & b_1 & \\ b_2 & h(B) & b_3 \\ & b_4 & \end{array} \right\rangle$ is the nth root of A, where $h(B) = \sqrt[n]{h(A)}$ and $b_i = \frac{a_i}{n\sqrt{h(A)}}, i =,1,...,4.$

6. Conclusion

In this paper we have developed necessary and sufficient conditions for the solvability of linear system over rhotrices if multiplication method proposed in [1] is used. These conditions depend on the heart of the respected rhotrices. We also devise an easy way of determining a square root and nth root of a rhotrix with positive heart.

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