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A remark on an arithmetic function. Part 2

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Let n be a fixed natural number and let $d_1, d_2, ..., d_s$ are all of its different divisors, ordered in a way that $n > d_1 > d_2 > ... > d_s > 1$. Therefore, the well-know arithmetic function σ has the form (see, e.g., [1])

$$\sigma(n) = n + d_1 + d_2 + ... + d_s + 1.$$

Here we shall introduce a new function that is somehow dual to function σ. It will have the form:

$$\chi(n) = n - d_1 + d_2 - ... + (-1)^s.d_s + (-1)^{s+1}.$$

Therefore, for each prime number p:

$$\chi(p) = p - 1.$$

It is appropriate to define that $\chi(1) = 0$.

On Table 1 the values of functions χ and σ for the first 50 natural numbers are given.

Table 1

n	$\chi(n)$	$\sigma(n)$	n	$\chi(n)$	$\sigma(n)$	n	$\chi(n)$	$\sigma(n)$
1	0	1	18	13	39	35	32	48
2	1	3	19	18	20	36	25	91
3	2	4	20	12	42	37	36	38
4	3	7	21	16	32	38	20	60
5	4	6	22	12	36	39	28	56
6	4	12	23	22	24	40	24	90
7	6	8	24	16	60	41	40	42
8	5	15	25	21	31	42	32	96
9	7	13	26	14	42	43	42	44
10	6	18	27	20	40	44	30	84
11	10	12	28	18	56	45	36	78
12	8	28	29	28	30	46	24	72
13	12	14	30	22	72	47	46	48
14	8	24	31	30	32	48	32	124
15	12	24	32	31	63	49	43	57
16	11	31	33	24	48	50	31	93
17	16	18	34	18	54			

It is easily proved the validity of the following assertions.

Proposition 1: For every natural number n

$$\frac{n}{2} < \chi(n) \leq n - 1.$$

Proposition 2: For every natural number $n = p.q$, where p, q are prime numbers and $p > q \geq 2$, the equality

$$\sigma(n) = \frac{q+1}{q-1}.\chi(n)$$

holds.

Really, if $n = p.q$ and $p > q \geq 2$, then

$$\sigma(n) = \sigma(p.q) = (p+1).(q+1) = \frac{q+1}{q-1}.(p+1).(q-1)$$

$$= \frac{q+1}{q-1}.(p.q - p + q - 1) = \frac{q+1}{q-1}.\chi(n).$$

Corollary 1: For every prime number $p \geq 3$:

$$\sigma(2p) = 3\chi(2p).$$

Corollary 2: For every prime number $p \geq 5$:

$$\sigma(3p) = 2\chi(3p).$$

Open problem 1: For every natural number n, $\chi(n) \geq \varphi(n)$, where φ is Euler's totient function (see, e.g. [1]).

Its more powerful form is

Open problem 2: For every natural number n,

$$n - \chi(n) \leq \varphi(n) \leq \chi(n).$$

References

[1] Nagell T., *Introduction to number theory,* John Wiley & Sons, New York, 1950.