NNTDM 15 (2009), 3, 14-20

THE INTEGER STRUCTURE OF THE DIFFERENCE OF TWO ODD-POWERED ODD INTEGERS

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Abstract

The modular ring Z_4 was used to analyse the structure of the integer, N, obtained from $x^n - y^n, x, y, n$ odd. The constraints on x and y associated with the probability of $x^n - y^n = N = z^n$ (z even) were explored. When $n \in \bar{3}_4$ (n = 3,7,11,15,…) the structure of N is $4r_0(4r_3 + 3)$, that is $\bar{0}_4 \times \bar{3}_4$. When $n \in \bar{1}_4$ (n = 5,9,13,17,…), the structure is $4r_0(4r_1 + 1)$, or $\bar{0}_4 \times \bar{1}_4$. The row structures and right-end-digit patterns of the rows of $(x^3 - y^3)$ and z^3 were compared and shown to be incompatible, as expected.

Keywords: primes, composites, modular rings, right-end digits, integer structure.

AMS Classification Numbers: 11A41, 11A07

1. Introduction

It is not yet widely realized that many complex theorems in number theory can be simplified by applying Integer Structure Theory (IST) [5]. As an example, we consider Fermat's Last Theorem for odd powers. The inequality

$$x^n - y^n \neq z^n, (n > 1; x, y, n, \text{ odd integers}) \tag{1.1}$$

has been established after many centuries of effort, with much laborious effort to find counter examples [2,3,11]. The aim here is to show how analysis with IST permits a high percentage of z values to be discarded. [1] contains references to the use of probability methods in number theory – a paper that was re-produced a year later by the Mathematical Association of New South Wales in their journal *Parabola.*

We use the modular ring Z_4 to describe the structure. We note that the recent work on generic frameworks of Műller-Olm and Seidl can be adapted for modular rings [10]. In this ring integers, N, can be represented by [5]:

$$N = 4r_i + \bar{i}, \tag{1.2}$$

in which r is the row in a modular array and $\bar{i}$ is the class. Class $\bar{2}_4$ integers are even and there are no powers in this class. Odd integers occur in classes $\bar{1}_4$ and $\bar{3}_4$ but there are no even powers in the latter class [5].

2. Constraints on *x* and *y*

The x and y components may fall in $\bar{1}_4$ or $\bar{3}_4$, and x^n, y^n will fall in the same class as x,y. Thus, with $x \in \bar{3}_4$ and $y \in \bar{1}_4$:

$$x^n - y^n = \bar{3}_4 - \bar{1}_4 = \bar{2}_4, \tag{2.1}$$

but class $\bar{2}_4$ has no powers, so this combination of x,y may be discarded. Similarly when $x \in \bar{1}_4$ and $y \in \bar{3}_4$:

$$x^n - y^n = \bar{1}_4 - \bar{3}_4 = \bar{2}_4. \tag{2.2}$$

Hence, x and y must fall in the same class; that is,

$$x^n - y^n = \bar{1}_4 - \bar{1}_4 = \bar{0}_4, \tag{2.3}$$

so that

$$x^n - y^n = 4r_0 \tag{2.4}$$

with

$$r_0 = R_1 - R_1^{'}. \tag{2.5}$$

Similarly,

$$x^n - y^n = \bar{3}_4 - \bar{3}_4 = \bar{0}_4, \tag{2.6}$$

and

$$x^n - y^n = 4\left(R_3 - R_3^{'}\right). \tag{2.7}$$

Since z^n has even rows, $R_1, R_1^{'}$ or $R_3, R_3^{'}$ must have the same parity.

3. Structure of $x^3 - y^3$

As shown previously [6],

$$x^n - y^n = (x-y)\left(x^{n-1} + y^{n-1} + xy\left(x^{n-2} - y^{n-2}\right)/(x-y)\right), \tag{3.1}$$

so

$$x^3 - y^3 = (x-y)\left(x^2 + y^2 + xy\right). \tag{3.2}$$

(i) Structure of $\left(x^2 + y^2 + xy\right)$.

(a) $x, y \in \bar{1}_4$:

$$x^2 + y^2 = \bar{1}_4 + \bar{1}_4 = \bar{2}_4 = 4R_2 + 2, \tag{3.3}$$

and

$$xy = \bar{1}_4 \times \bar{1}_4 = 4R_1 + 1 \tag{3.4}$$

so that

$$x^2 + y^2 + xy = 4(R_2 + R_1) + 3. \tag{3.5}$$

Thus this factor falls in class $\bar{3}_4$.

(b) $x, y \in \bar{3}_4$:

$$x^2 + y^2 = \bar{1}_4 + \bar{1}_4 = \bar{2}_4 = 4R_2' + 2, \tag{3.6}$$

since there are no even powers in $\bar{3}_4$, and

$$xy = \bar{3}_4 \times \bar{3}_4 = \bar{1}_4 = 4R_1' + 1 \tag{3.7}$$

so that

$$x^2 + y^2 + xy = 4(R_2' + R_1') + 3. \tag{3.8}$$

Again this factor falls in class $\bar{3}_4$.

(ii) Total structure of $x^3 - y^3$.

Since $(x-y) \in 4r_0$ always, the total structure has the form:

$$x^3 - y^3 = 4r_0\left(4r_3 + 3\right) \tag{3.9}$$

which means that z^3 cannot be of the form $\left(2^m\right)^3$ as a factor in $\bar{3}_4$ must be present. Thus all even integers of the forms $\left(2^m\right)$ or $\left(2^m\left(4R_1 + 1\right)\right)$ can be discarded. However, if r_0 has an odd number of factors in $\bar{3}_4$, then this would mask $\left(4r_3 + 3\right)$ because an even product of $\bar{3}_4$ integers falls in $\bar{1}_4$. A check on (x-y) will eliminate this problem.

Obviously, these results put severe limitations on possible z values that might produce counter examples to (1.1).

(iii) Examples of row structures for $x^3 - y^3$ and z^3.

Table 1 shows the row structures for all combinations of $x^3 - y^3$, $x, y \in \bar{1}_4, \bar{1}_4$ for the first 100 integers. Since the row of z^3 must be even, half the combinations (those with odd rows) may be discarded. The $x^3 - y^3$ rows are compared with the row structures of z^3 (*z* even, using the permissible values of *z*). The over-bars and subscripts (always 4) have been omitted for simplicity. Thus, 212010111 or $\bar{2}_4\bar{1}_4\bar{2}_4\bar{0}_4\bar{1}_4\bar{0}_4\bar{1}_4\bar{1}_4\bar{1}_4$ indicates that the row of $x^3 - y^3$ is in class $\bar{2}_4$, the row of this row is in class $\bar{1}_4$, and so on. Commonly z^3 has one of the row structures $\bar{0}_4\bar{0}_4\ldots, \bar{2}_4\bar{1}_4\ldots, \bar{2}_4\bar{3}_4\ldots$.

$x^3 - y^3$	z^3	$x^3 - y^3$	z^3	$x^3 - y^3$	z^3
	000000213	031231112		212232222	
	0002112122	03200023		21231212	
	00220321	032211		212321	
		032302			
003020002	00303011	032321101		21311222	213323011
003232321	003111	032333		213122	
	00312132131	033233222		21323321	
	00332013	200101221		2133100	
		20031012			
01011012		201022311		220301321	
012013212		201103202			
01200132		20121231		22113022	
0123302		2021202			
02102321		203231		223021202	
0211022		21001331	2103121		
0211121031		21002		223231331	
021210001				230000321	23003311
02121122		210323		230213302	
0301231		21100	2111030001		
03020130		211211112	2112113		231032203
03021031		2112332			23112221
030223		211330			
03032233		212010222	212013212	23202031	232133031
03033301		21203021		23303302	
		2121023	212122	233223331	23320223

Table 1: Row Structures (bars and suffixes omitted for simplicity)

A similar table may be constructed for $x, y \in \bar{3}_4, \bar{3}_4$. Since, for a cube, $n \in \bar{3}_4$ and when n is in this class the structure remains of the form $4r_0(4r_3+3)$. However, when $n \in \bar{1}_4$ (5,9,13,…), the structure becomes $4r_0(4r_1+1)$. This is illustrated in Section 4 for $n = 5$.

When r_0 from Equation (3.9) falls in $\bar{2}_4, (4r_2+2)$, the row of x^3-y^3 is given by

$$4(4r_2r_3+3r_2+2r_3+1)+2 \tag{3.10}$$

and hence falls in class $\bar{2}_4$. The row of the row will fall in classes that depend on the parity of r_2. If $r_0 \in \bar{0}_4$, then the row falls in $\bar{0}_4$:

$$4(4r_0'r_3+3r_0'). \tag{3.11}$$

The row of the row depends on the parity of r_0'.

(iv) Right-end-digit structure of rows.

Table 1 shows that relatively few row structures for x^3-y^3 and z^3 appear to be similar. These similar groups may be compared on the basis of the right-end-digit (RED) structures of these row sequences. Such RED structures have been previously used for an analysis of primes [4,8], Pythagorean triple analysis [9], and products of adjacent integers [7]. The comparison is illustrated in Table 2. The apparent similarity of the row structures does not hold with this further criterion.

x^3-y^3		z^3	
Rows	REDs	Rows	REDs
		00220321	82848761
003020002	287648282	00301311	28769315
		0030301	64123051
		003111	823515
		00312132131	28756158971
		003203222	46948766
		00332013	00530233
21001331	61005171	21031212	61053892
21002	89282		
210323	256943		
21100	61564		
		2111030001	2515694641
211211112	251079792	2112113	8974333
2112332	2510712		

211330	897994		
212010222	610238202	212013212	438415892
21203021	25669461	212122	256102
2121023	8925643		
21311222	25179202	213323011	07994305
213122	897102		
2132332	89761761		
2133100	6153564		
23000032	07646412	23003311	43005351
2302031	0766487		
2320203	07664871	232133031	894335871
23303303	071207171	23320223	25302843
233303303	07123582		

Table 2: RED sequences for rows of $x^3 - y^3$ that are similar to z^3

Obviously, the difference of two odd cubes produces an integer that cannot follow the structure required for an even cube. This is, of course, to be expected from the theorem.

4. Structure of $x^5 - y^5$

The constraints on *x*,*y* remain the same. However, the factor

$$\left(x^{n-1} + y^{n-1} + \frac{xy}{x-y}\left(x^{n-2} - y^{n-2}\right)\right)$$

will have a different structure from systems where $n \in \bar{3}_4$. For n = 5 the factor F becomes:

$$\begin{aligned} F &= \left(\mathrm{x}^4 + y^4 + \tfrac{xy}{x-y}\left(x^3 - y^3\right)\right) \\ &= x^4 + y^4 + xy\left(x^2 + y^2 + xy\right) \end{aligned} \tag{4.1}$$

from Equation (3.2). As for n =3,

$$x^4 + y^4 = \bar{1}_4 + \bar{1}_4 = \bar{2}_4 \tag{4.2}$$

$$xy = \bar{1}_4 \times \bar{1}_4 = \bar{1}_4 \tag{4.3}$$

$$x^2 + y^2 + xy = \bar{3}_4\,. \tag{4.4}$$

Thus,

$$F = \bar{2}_4 + \bar{3}_4 = \bar{1}_4 \tag{4.5}$$

so that the structure of $x^5 - y^5$ when $x, y \in \left\{\bar{1}_4\right\}$ is given by

$$x^5 - y^5 = 4r_0\left(4r_1 + 1\right). \tag{4.6}$$

The same structure is obtained when $x, y \in \{\bar{3}_4\}$, since $\bar{3}_4 \times \bar{3}_4 \in \bar{1}_4$ and all even powers of odd integers fall in class $\bar{1}_4$. Hence, the constraints on *z* will be that r_0 is even , with $(2^m)^n$ invalid, and the odd factor must be in class $\bar{1}_4$ for z^5. The same structure is obtained for all $n \in \bar{1}_4$.

References

1. Clarke, J.H., A.G. Shannon.1983. A combinatorial approach to Goldbach's conjecture. *Mathematical Gazette*. 67: 44–46.
2. Cornell, G., J.H. Silverman, G. Stevens (eds.) 1997. *Modular Forms and Fermat's Last Theorem*. Berlin: Springer.
3. Faltings, Gerd. 1995. The Proof of Fermat's Last Theorem by R. Taylor and A. Wiles. *Notices of the American Mathematical Society*. 42: 743-746.
4. Leyendekkers, J.V., A.G. Shannon. (In press) Analysis of Primes Using REDs (Right-End-Digits) and Integer Structure. *Notes on Number Theory & Discrete Mathematics*.
5. Leyendekkers, J.V., A.G. Shannon, J.M. Rybak. 2007. *Pattern Recognition: Modular Rings and Integer Structure.* North Sydney: Raffles KvB Monograph No.9.
6. Leyendekkers, J.V., A.G. Shannon. 2007. Modular Ring Class Structures of $x^n \pm y^n$. *Notes on Number Theory & Discrete Mathematics*. 13(3): 27-35.
7. Leyendekkers, J.V., A.G. Shannon, C.K. Wong. 2008. Integer Structure Analysis of the Product of Adjacent Integers and Euler's Extension of Fermat's Last Theorem. *Advanced Studies in Contemporary Mathematics*. 17: 221-229.
8. Leyendekkers, J.V., A.G. Shannon, C.K. Wong. 2009. The Spectra of Primes. *Proceedings of the Jangjeon Mathematical Society*. 12: 1-10.
9. Leyendekkers, J.V., A.G. Shannon, C.K. Wong. (In press). Structure and Spectra of the Components of of pPts and Fermat's last Theorem. *Notes on Number Theory & Discrete Mathematics*.
10. Műller-Olm, Markus, Helmut Seidl. 2005. A Generic Framework for Interprocedural Analysis of Numerical Properties. In Chris Hankin, Igor Silveroni (eds). *Static Analysis*. Berlin: Springer, pp.235-250.
11. Wiles, A. 1995. Modular Elliptic Curves and Fermat's Last Theorem. *Annals of Mathematics*. 141: 443-551.