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THE IDENTIFICATION OF ROWS OF PRIMES IN THE MODULAR RING Z_6

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Abstract

The simple function $f(n)=\frac{1}{2}n(an\pm 1), a=1,3,5$ with n = 1,2,…,200, generated 615 primes in the modular ring Z_6. 194 of these were twin primes. Values of n which yielded primes for all $f(n)$ were simply related to the number of primes in a given range.

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1. Introduction

We have recently shown [4] that the simple function

$$f(n)=\tfrac{1}{2}n(an\pm 1) \tag{1.1}$$

with a =1,3,5, commonly predicts those rows which contain primes in the modular ring Z_6. To simplify the analysis, we used the right-end-digit (RED) criteria [4]; that is, odd integers were separated into N^* = 1,3,7,9 where the asterisk denotes the RED. For 80 n values, about 500 primes were generated for RED = 7 alone. Similar results were obtained for the other REDs.

Here we continue to analyse which values of a in Equation (1.1) produce rows which contain primes in Z_6. This modular ring has primes identified by $(6r_i+(i-3))$ in which r is the row and $\bar{i}$ is the class as in Table 1. The primes are represented by $(6r\pm 1)$. Hence, with $r=f(n)$, the a values which yield primes may be determined.

Table 1

Class		$\bar{1}_6$	$\bar{2}_6$	$\bar{3}_6$	$\bar{4}_6$	$\bar{5}_6$	$\bar{6}_6$
Row	0	-2	-1	0	1	2	3
	1	4	5	6	7	8	9
	2	10	11	12	13	14	15
	3	16	17	18	19	20	21
	4	22	23	24	25	26	27

2. Calculation of *a* values which yield rows of primes

Equation (1.1) was used to calculate the rows of integers in Z_6 for n =1 to 200.

For n = 1 to 100, Table 2 shows which values of a produced prime-rows and the class of the prime. The class of the prime is indicated by the subscripts 2 and 4. These are almost 100 of the possible 16,384 combinations. All n values in Table 2 produced primes for one or more values of a, an exception being n = 43. However, if $a = 7$ is used for n = 43, the primes 38959 ($\bar{4}_6$) and 38699 ($\bar{2}_6$) are produced. Some n values produced primes for all values of a (1,3,5).

For n = 101 to 200, primes were produced for 91% of the n, exceptions being n = 106,122,148,166,181,188,192,199 and 200. However, if larger values of a are used (say 7,9), then primes are produced. For example, for n = 106 with a = 9, primes 303689,303691 and 303053 are produced, while for n = 122, $\frac{1}{2}n(7n-1)$ produces primes in classes $\bar{2}_6$ and $\bar{4}_6$. Obviously, the a-function needs to be developed further for a general solution.

For the 200 n values, 615 primes were generated with a = 1,3,5. The n values which generated primes for all $f(n)$ could be expected to be related to the number of primes, Δ, in the $f(n)$ range (a = 1,3,5). This is evident in Table 3 where Δ/n is closely related to n, and is almost linear for $n \geq 30$.

Table 2: $f(n)$ code

n	**$f(n)$ = prime row**	**n**	**$f(n)$ = prime row**	**n**	**$f(n)$ = prime row**
1	$A_2A_4, B_2B_4, C_2C_4, D_2D_4, E_2E_4$	34	$A_4, -, C_2C_4, D_4, E_4$	68	$-, -, C_2C_4, -, -$
2	$A_2A_4, B_2B_4, C_2C_4, D_4, E_2$	35	$A_2, B_4, -, D_4, E_2$	69	$A_2, -, C_2C_4, -, E_4$
3	$A_4, B_2, C_2C_4, -, E_4$	36	$-, -, -, -, E_4$	70	$-, -, C_2C_4, D_2, E_4$
4	$A_2A_4, B_4, C_2, D_2, E_2E_4$	37	$A_2A_4, B_4, C_4, -, -$	71	$-, -, -, -, E_2E_4$
5	$A_2, B_2B_4, C_4, D_2, E_2$	38	$A_4, B_2, -, D_2, -$	72	$A_2, -, C_2C_4, D_4, E_2$
6	$A_4, -, C_4, D_2, E_2E_4$	39	$A_2, B_4, -, -, E_2E_4$	73	$-, B_2, C_2C_4, D_2, -$
7	$A_2, B_2B_4, C_2C_4, D_4, -$	40	A_2, B_2, C_4, D_4, E_2	74	$A_2A_4, -, -, D_2, E_4$
8	$-, B_2B_4, -, D_2, E_4$	41	$A_4, -, -, D_4, -$	75	$A_2, B_2, -, -, -$
9	$A_2A_4, B_4, C_2, -, E_2$	42	$A_2A_4, B_2, C_2, -, -$	76	$-, -, -, D_4, E_4$
10	$A_4, B_2, -, D_4, E_4$	43	$-, -, -, -, -$	77	$-, B_2B_4, C_2, -$
11	$A_4, B_4, -, D_2, E_4$	44	$A_2, -, C_2C_4, D_4, E_4$	78	$-, -, C_2, D_2, -$
12	$A_2, -, C_2, -, -$	45	$A_4, -, C_2, D_2, E_4$	79	$A_2, -, C_2C_4, D_2, E_2$
13	$A_4, B_2, C_2C_4, -, -$	46	$-, B_2B_4, -, -, E_2$	80	$A_4, B_2, -, -, -$
14	$A_4, -, C_2C_4, -, E_2$	47	$-, B_2B_4, C_2, -, E_2$	81	$A_4, -, -, -, -$
15	$A_2, B_2, C_2, -, E_2E_4$	48	$A_4, B_2, C_4, D_2, -$	82	$-, B_2B_4, C_4, D_4, E_2$
16	$-, B_2, -, D_4, E_4$	49	$A_2A_4, B_4, -, D_2, E_4$	83	$-, -, C_2, D_2, E_4$
17	$A_4, -, C_2C_4, -, E_2$	50	$A_2, B_4, C_2, D_2, -$	84	$A_2, -, -, -, -$
18	$-, B_2B_4, C_2, -, -$	51	$-, B_2B_4, -, -, E_2$	85	$A_2, -, -, D_4, -$
19	$-, B_4, C_2, D_2, -$	52	$A_4, -, C_2C_4, -, -$	86	$A_4, B_2, -, -, E_2$
20	$A_2, B_2, C_2C_4, -, E_2$	53	$-, B_2, C_2, D_2, -$	87	$-, -, -, D_4, -$
21	$-, -, C_4, D_4, E_2E_4$	54	$-, B_4, C_4, -, E_2E_4$	88	$A_4, B_2, C_2, D_2, -$
22	$-, B_2B_4, C_2, -, E_2$	55	$A_2A_4, -, C_2C_4, D_4, -$	89	$A_2, -, C_4, D_4, E_4$
23	$A_4, B_4, C_2, -, E_4$	56	$-, B_4, C_4, D_2, -$	90	$A_4, -, -, -, E_2$
24	$A_4, -, C_4, D_4, -$	57	$-, B_2, -, D_4, E_2$	91	$A_4, -, C_4, D_4, E_2$
25	$A_2A_4, B_4, -, -, -$	58	$A_4, B_2, C_4, -, E_4$	92	$A_2, -, -, -, E_2$
26	$-, B_4, C_4, -, E_2$	59	$-, -, C_2C_4, D_2, -$	93	$A_4, B_4, C_4, -, E_4$
27	$A_2A_4, -, C_4, -, E_2$	60	$A_2, B_2, -, D_4, E_2$	94	$-, -, C_2, -, E_2$
28	$A_4, -, C_2, -, E_4$	61	$-, -, -, D_2, E_2E_4$	95	$A_4, B_2, -, D_4, E_2$
29	$A_2, -, C_2, D_4, E_2$	62	$A_2A_4, B_2, -, -, -$	96	$-, B_2B_4, C_4, -, -$
30	$A_2, B_4, C_2C_4, D_4, E_4$	63	$A_4, B_4, C_2C_4, -, -$	97	$A_2, -, C_2C_4, -, -$
31	$-, B_2, -, -, E_2E_4$	64	$A_2, B_4, C_2, D_2, -$	98	$-, B_2, C_4, -, -$
32	$A_2A_4, B_2, -, -, E_2$	65	$-, B_2, C_4, -, E_2$	99	$-, -, C_2, D_2, E_4$
33	$-, B_4, -, D_2, -$	66	$A_4, -, -, D_2D_4, E_2$	100	$-, -, -, D_2D_4, -$
		67	$A_4, -, -, D_4, -$		

Legend: $A = \frac{1}{2}n(n+1); B = \frac{1}{2}n(3n+1); C = \frac{1}{2}(3n-1); D = \frac{1}{2}n(5n+1); D = \frac{1}{2}n(5n-1)$.

Subscripts: $2 \in \overline{2}_6; 4 \in \overline{4}_6$.

Table 3: Values of *n* which yield primes for all *f*(*n*), *a* = 1,3,5

n	Primes from *f*(*n*) = row					Δ=number of primes *D-A*	$\frac{\Delta}{n}$
	A	*B*	*C*	*D*	*E*		
2	17 19	41 43	29 31	67	53	16	8
4	59 61	157	131 133	251	227 229	38	9.5
5	89	239 241	211	389	359	54	10.8
30	2789	8191	8009 8011	13591	13411	1203	40.1
40	4919	14519	14281	24121	23879	2030	50.8
134	54269	162007	161201	269741	268937	17755	132.5
165	82171	245519 245521	244529	408869	407879	26506	160.6
191	110017	328901	327757	547787	546643	35216	184.4

See Table 2 for the *f*(*n*) code

Certain *n* values have REDs which produce integers with a RED of 5, which, of course, cannot be a prime when $N > 5$. For example, when

$$(f(n))^* = 6,\ (6\text{x}6\text{-}1)^* = 5, \text{ or } (f(n))^* = 4,\ (6\text{x}4+1)^* = 5,$$

as in Table 4. Obviously, $f(n) = \frac{1}{2}n(3n \pm 1)$ potentially has a higher yield of primes.

Table 4: n^* for which $N^* = 5$

f(*n*)	$N \in \overline{2}_6$	$N \in \overline{4}_6$
$\frac{1}{2}(n+1)$	1,3,6,8	Nil
$\frac{1}{2}n(3n+1)$	4,9	Nil
$\frac{1}{2}n(3n-1)$	1,6	Nil
$\frac{1}{2}n(5n+1)$	2,7	3,8
$\frac{1}{2}n(5n-1)$	3,8	2,7

Twin Primes

For *a* = 1,3,5, 194 twin primes were generated for *n* = 1 to 200. Values of *n* for the various *f*(*n*) which yield twin primes are listed in Table 5. The *f*(*n*) yield the rows so that the twin primes are obtained from $(6f(n) \pm 1)$.

Table 5: Twin Primes

$f(n)$	***n* values which yield twin primes**
$A=\frac{1}{2}n(n+1)$	1,2,4,9,25,27,32,37,42,49,55,62,74,119,140,142,147,175,195
$B=\frac{1}{2}n(3n+1)$	1,2,5,7,8,18,22,46,47,51,77,82,96,103,117,126,135,151,152,165
$C=\frac{1}{2}(3n-1)$	1,2,3,7,13,14,17,20,30,34,44,52,55,59,63,68,69,70,72,73,79,97,105, 132,152,167,173,177,184,187
$D=\frac{1}{2}n(5n+1)$	1,66,100,121,151,156,160,161,170,184,195
$D=\frac{1}{2}n(5n-1)$	1,4,6,15,21,31,39,54,61,71,109,116,149,160,164,185

Z_6 rows of twin primes have been analysed previously [1,2] and a method for predicting them is now outlined. While many twin-prime rows are predicted by $f(n)$ with a = 1,3,5, a more general function $\frac{1}{2}n(an\pm b)$ seems necessary for predicting all the rows. For example, b = 3 gave many more rows.

Table 4 sets out the n* values which produce N* = 5, so for twin primes none of the n values with REDs listed could give twin primes, whether in $\bar{2}_6$ or $\bar{4}_6$, as one of the integers produced would have a RED of 5 and hence could not be a prime. This is evident in Table 5 and explains why D and E produce far fewer twin primes. However, it is not clear why C produces the most.

Brun's Theorem states that the sum of the reciprocals of twin primes is convergent with a finite value known as Brun's constant [5]. It has historical importance in the introduction of sieve methods. Its importance here is that this result shows that the sum of the reciprocals of the twin primes converges; in other words the p involved are a small set. In explicit terms the sum

$$\sum_{p:\,p+2\in\mathbb{P}}\left(\frac{1}{p}+\frac{1}{p+2}\right)=\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\cdots$$

either has finitely many terms or has infinitely many terms but is convergent: its value is known as Brun's constant. Unlike the case for all prime numbers, we cannot conclude from this result that there are an infinite number of twin primes. In our case,

$$B=\left(\tfrac{1}{3}+\tfrac{1}{5}\right)+\left(\tfrac{1}{5}+\tfrac{1}{7}\right)+\left(\tfrac{1}{11}+\tfrac{1}{13}\right)+\ldots\approx 1.902. \tag{3.1}$$

Since in Z_6 primes are given by $(6r\pm 1)$, Equation (3.1) may be expressed by

$$B=0.533+12\sum_{r=1}^{\infty}\frac{r}{36r^2-1}\approx 1.9022$$

in which r is the specific row which contains the twin primes, so that

$$\sum_{r=1}^{\infty}\frac{r}{36r^2-1}\approx 0.11407\approx\frac{e}{24}.$$

This additional information might be of use in developing the general equation for the row; that is, $r=\frac{1}{2}n(an\pm b)$.

4. Final Comments

The triangular and pentagonal numbers are proportional to the rows of squares in modular rings [3], so that the compatibility of these simple functions with prime-rows is somewhat surprising. However, the forms are similar to the Euler equation

$$N = x^2 + x + p \tag{4.1}$$

where $p = 2,3,5,11,17,41$ yields primes for $0 \le x \le p-2$.

It has also been shown [3] that Equation (4.1) is compatible with integer structure and Z_6. An extended form of Euler's equation may be used up to very high integer values [3].

Another interesting feature of the present analysis is that n values that produce primes for all $f(n)$ are closely related to the number of primes in a given range. The sequence of these types of n values could be used to investigate prime distribution.

References

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