NNTDM 14 (2008) 4, 1-3

A short remark on transcendental numbers

Mladen Vassilev - Missana
5, V. Hugo Str., Sofia 1124, Bulgaria
e-mail: *missana@abv.bg*

Abstract. In the paper an application of the famous Gelfond-Schneider theorem is made.

Below the following variant of Gelfond-Schneider theorem (see [1]) shall be used:

Theorem 1. If a $(a \neq 0, 1)$ is an algebraic number and b is an irrational algebraic number, then a^b is a transcendental number.

Further we shall use the denotation $\sqrt[x]{x}$ for $x^{\frac{1}{x}}$ and as usual e for John Napier's number: $e = 2.71828....$

let us observe that $\sqrt[e]{e} = 1.44466....$

Our first result in the present paper is

Theorem 2. Let a be an algebraic number such that

$$1 < a < \sqrt[e]{e} \tag{1}$$

holds and

$$a = \sqrt[b]{b} \tag{2}$$

does not hold for any rational number $b > 1$. Then:

1) The equality

$$a^x = x \tag{3}$$

has exactly two different solutions: $x_1 \in (1, e)$ and $x_2 \in (e, +\infty)$;

2) x_i $(i = 1, 2)$ are transcendental numbers;

3) The algebraic number a admits the following two different representations, using the transcendental numbers x_1 and x_2:

$$a = \sqrt[x_1]{x_1} \text{ and } a = \sqrt[x_2]{x_2}\ . \tag{4}$$

Proof. First, we rewrite (3) in the form

$$\sqrt[x]{x} = a. \tag{5}$$

Second, we introduce a function $f : (1, +\infty) \to (1, \sqrt[e]{e})$ putting

$$f(x) = \sqrt[x]{x}.$$

It is clear that f is a continuous function strictly increasing on interval $(1, e)$ and strictly decreasing on interval $(e, +\infty)$. Also, f has an absolute maximum at $x = e$. Therefore, for $x \in (1, e)$ we have:

$$1 = f(1) < f(x) < f(e) = \sqrt[e]{e}$$

and for $x \in (e, +\infty)$ we have

$$\sqrt[e]{e} = f(e) > f(x) > 1 = \lim_{x \to +\infty} f(x).$$

Hence 1) holds, because of (1) and (5).

Let $x = b$ be any solution of (3). Then $x = b$ is a solution of (5), too. Therefore, b satisfies (2). Hence b is an irrational number.

Let us assume that b be an algebraic number. Then Theorem 1 yields that a^b is a transcendental number. But

$$a^b = a$$

since $x = b$ is a solution of (3). Hence a is a transcendental number. The last contradicts to the fact that a is an algebraic number. Therefore, our assumption that b is an algebraic number is wrong. Hence b is a transcendental number and 2) is proved.

Now, 3) (in particular (4)) holds from 1) and 2).

The Theorem is proved.

From every $a \in [1, \sqrt[e]{e}]$ we introduce a infinite sequence $K_1(a), K_2(a), K_3(a), \ldots$, putting

$$K_1(a) = a,$$

$$K_{n+1}(a) = a^{K_n(a)}, \text{ for } n \geq 1. \tag{6}$$

The above sequence is stricly increasing and bounded. The last fact follows from the inequality

$$K_n(a) < K_n(\sqrt[e]{e})$$

that is valid for $n \geq 1$. Indeed, it is a matter of check that

$$\lim_{n \to +\infty} K_n(\sqrt[e]{e}) = e.$$

Hence

$$K_n(a) < e$$

for $n \geq 1$.

Therefore, $K_n(a)$ converges to the finite limit $K(a)$ that we denote by

$$K(a) = a^{a^{a^{\cdot^{\cdot}}}}.$$

If we put

$$K(a) = x_1,$$

then x_1 satisfies (3), because of (6).

Thus as a corollary of Theorem 2 we obtain

Theorem 3 Let a be an algebraic number such that (1) holds and (2) does not hold for any rational number $b > 1$. Then, there exists

$$x_1 = \lim_{n\to+\infty} K_n(a) \equiv a^{a^{a^{\cdot^{\cdot}}}}$$

and $x_1 \in (1, e)$ is a transcendental number.

If we take $a = \sqrt[b]{b}$ with $b \in [1, e]$ (this time without the restriction for b to be irrational number), then it is easy to see that

$$\lim_{n\to+\infty} K_n(a) = b.$$

For example, if $b = 2$ or 4, we have

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}} = 2.$$

$$\sqrt[4]{4}^{\sqrt[4]{4}^{\sqrt[4]{4}^{\cdot^{\cdot}}}} = 2.$$

On the other hand

$$\sqrt[3]{3}^{\sqrt[3]{3}^{\sqrt[3]{3}^{\cdot^{\cdot}}}}$$

is a transcendental number. Sinse the Diophantive equation

$$\sqrt[a]{a} = \sqrt[b]{b} \tag{7}$$

with $a \neq b$ has an integer solutions only in the case $a = 2$, $b = 4$ then we conclude that for each natural number $n > 1$ and $n \neq 2, 4$

$$\sqrt[n]{n}^{\sqrt[n]{n}^{\sqrt[n]{n}^{\cdot^{\cdot}}}}$$

is a transcendental number.

Open problem: To be described all rational numbers a, b such that $a \in (1, e), b \in (e, +\infty)$ which are solutions of (7).

References

[1] Baker, A. Transcendental Number Theory. London, Cambridge University Press, 1990.