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THREE-DIMENSIONAL EXTENSIONS OF FIBONACCI SEQUENCES. Part 1

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1 Introduction

The Fibonacci sequence is an object of a lot of generalizations. During the last 25 years the author took part in introduction of some of them, like 2-Fibonacci sequences (together with D. Sasselov and my wife L. Atanassova), sets of Fibonacci sequences (together with A. Shannon and my daughter V. Atanassova), Fibonacci plane, Fibonacci space (together with A. Shannon) and others.

In all cases the new sequences are planar objects. Even each of the sequences in the Fibonacci space has a line-form, i.e., planar form.

Now, in a series of papers, we shall introduce the first three-dimensional Fibonacci-like objects. In the present (first) part we shall construct the first two sequences with three-dimensional form that can be project in a plane.

2 Array of four three-dimensional Fibonacci sequences

Let us have eight fixed real number $a_1, ..., a_8$ and let there be a parallelepiped with square-basis that is the first level of the figure. Let this basis contains four (planar) squares in which numbers $a_1, ..., a_4$ stay. This first level has the form:

a_1	a_2
a_4	a_3

first level

Let in the second level of the figure (containing also four squares) numbers $a_5, ..., a_8$ stay. The second level has the form:

a_5	a_6
a_8	a_7

second level

Now, the elements of the third level of the parallelepiped will be obtained from the first eight elements, as follows:

$a_3 + a_8$	$a_4 + a_5$
$a_2 + a_7$	$a_1 + a_6$

third level

The four next levels have, respectively, the forms:

$a_2 + 2a_7$	$a_3 + 2a_8$
$a_1 + 2a_6$	$a_4 + 2a_5$

fourth level

$2a_1 + 3a_6$	$2a_2 + 3a_7$
$2a_4 + 3a_5$	$2a_3 + 3a_8$

fifth level

$3a_4 + 5a_5$	$3a_1 + 5a_6$
$3a_3 + 5a_8$	$3a_2 + 5a_7$

sixth level

$5a_3 + 8a_8$	$5a_4 + 8a_5$
$5a_2 + 5a_7$	$5a_1 + 5a_6$

seventh level

Let the arithmetic function φ be defined as follows for each natural number n:

$$\varphi(n) = m \text{ if and only if } n \equiv m (\text{mod} 4) \text{ and } 1 \leq m \leq 4.$$

If the four sequences are $\{\alpha_{i,j}\}_{j=0}^{\infty}$ $(1 \leq i \leq 4)$, then they can be defined as

$$\alpha_{1,0} = a_1, \alpha_{1,1} = a_5, \alpha_{1,n+2} = \alpha_{4,n+1} + \alpha_{3,n},$$

$$\alpha_{2,0} = a_2, \alpha_{2,1} = a_6, \alpha_{2,n+2} = \alpha_{1,n+1} + \alpha_{4,n},$$

$$\alpha_{3,0} = a_3, \alpha_{3,1} = a_7, \alpha_{3,n+2} = \alpha_{2,n+1} + \alpha_{1,n},$$

$$\alpha_{4,0} = a_4, \alpha_{4,1} = a_8, \alpha_{4,n+2} = \alpha_{3,n+1} + \alpha_{2,n}.$$

Let $\{F_j\}_{j=0}^{\infty}$be the ordinary Fibonacci sequence $0, 1, 1, 2, ...$ The following assertion can be proved, e.g., by induction.

Theorem 1: For every natural numbers i, j, k such that $1 \leq i, j \leq 4:$

$$\alpha_{i,4k+j} = F_{4k+j-1} a_{\varphi(i+4-j)} + F_{4k+j} a_{\varphi(i+5-j)+4}.$$

3 Set of extended Fibonacci sequences

Let the real numbers a, b_1, b_2, b_3, b_4 be given. Then we can construct the following three-dimensional figure that will correspond to a set of sequences and that will have as first four levels the following squares:

a

first level

b_1	b_2
b_4	b_3

second level

$a+b_1$	$a+b_2$	
$a+b_4$	$a+b_3$	third level

$a+2b_1$	$a+b_1+b_2$	$a+b_1+b_2$	$a+2b_2$	
$a+b_1+b_4$	$a+b_1+b_3$	$a+b_2+b_4$	$a+b_2+b_3$	fourth level
$a+b_1+b_4$	$a+b_2+b_4$	$a+b_1+b_3$	$a+b_2+b_3$	
$a+2b_4$	$a+b_3+b_4$	$a+b_3+b_4$	$a+2b_3$	

It can be seen directly that if C_i is the number of members of the square from the i-th level ($i = 0, 1, 2, ...$), then these numbers can be expressed in terms of elements of the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$:

$$C_0 = 1 = (2^{F_0})^2,\ C_1 = 4 = (2^{F_1})^2,\ C_2 = 4 = (2^{F_2})^2,\ C_3 = 16 = (2^{F_3})^2,\ ...$$

By induction we can prove

Theorem 2: The number of the elements of the square from i-th level ($i = 0, 1, 2, ...$) is

$$C_i = (2^{F_i})^2 = 2^{2F_i}.$$

This Fibonacci extension is based on the extension from [1] that corresponds to the planar case, while the above one is a non-planar case.

References

[1] Atanassova V., A. Shannon, K. Atanassov, Sets of extensions of the Fibonacci sequences. *Comptes Rendus de l'Academie bulgare des Sciences,* Tome 56, 2003, No. 9, 9-12.