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ON AN EXTREMAL PROBLEM RELATED TO THE DELANNOY NUMBERS

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ABSTRACT: In the paper the maximal elements of the set $\{D(n-k,k)|k = 0,1,...,n\}$ are found, where $D(p,q)$ are the so-called Delannoy numbers and $n \geq 2$ is a natural number. It is shown that for an even n the number $D(\frac{n}{2},\frac{n}{2})$ is the maximal element of the mentioned set, while when n is odd – the maximal elements are two $D([\frac{n}{2}]+1,[\frac{n}{2}])$ and $D([\frac{n}{2}],[\frac{n}{2}]+1)$.

It is well known that Henri Delannoy's numbers $D(p,q)$ satisfy the following three relations (see [1, 2]):

$$D(p,q) = D(q,p), \tag{1}$$

$$D(p,q) = \sum_{j\geq 0} 2^j.\binom{p}{j}.\binom{q}{j}, \tag{2}$$

$$D(p,q) = \sum_{j\geq 0} \binom{q}{j}.\binom{p+q-j}{j}. \tag{3}$$

Further, we will not use (2), but only (1) and (3).

Lemma 1. Let $n \geq 2$. If

$$k \leq \frac{n}{2} - 1,$$

then the relation

$$D(n-k-1,k+1) = D(n-k,k) + \delta(n,k) \tag{4}$$

holds, where

$$\delta(n,k) = \binom{n-k-1}{k+1} + \sum_{j=1}^{k} \binom{k}{j-1}.\binom{n-j}{j}.$$

Proof. Putting in (3) $q = k+1$, $p = n-k-1$, we obtain

$$D(n-k-1,k+1) = \sum_{j\geq 0} \binom{k+1}{j}.\binom{n-j}{j}.$$

Since

$$k \leq \frac{n}{2} - 1,$$

by condition, the above equality yields

$$D(n-k-1,k+1)=\sum_{j=0}^{k=1}\left(\begin{array}{c}k+1\\ j\end{array}\right).\left(\begin{array}{c}n-j\\ j\end{array}\right). \quad (5)$$

Since for $j \geq 1$ we have

$$\left(\begin{array}{c}k+1\\ j\end{array}\right)=\left(\begin{array}{c}k\\ j\end{array}\right)+\left(\begin{array}{c}k\\ j-1\end{array}\right),$$

after an elementary computation we obtain

$$D(n-k-1,k+1)=\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right).\left(\begin{array}{c}n-j\\ j\end{array}\right)+\delta(n,k). \quad (6)$$

From (3) with $q=k$, $p=n-k$, we have

$$\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right).\left(\begin{array}{c}n-j\\ j\end{array}\right)=\sum_{j\geq 0}\left(\begin{array}{c}k\\ j\end{array}\right).\left(\begin{array}{c}n-j\\ j\end{array}\right)=D(n-k,k). \quad (7)$$

Hence, (4) holds, because of (6) and (7).

Lemma 1 is proved.

Corollary 1. Let $n \geq 2$. If

$$k \leq \frac{n}{2}-1,$$

then the inequality

$$D(n-k,k)<D(n-k-1,k+1) \quad (8)$$

holds.

Lemma 2. Let $n \geq 2$ and $k \geq \frac{n}{2}$. Then the inequality

$$D(n-k-1,k+1)<D(n-k,k) \quad (9)$$

holds.

Proof. Since $n \geq 2$ and $k \geq \frac{n}{2}$ (by condition) putting

$$m=n-k-1, \quad (10)$$

we obtain

$$m<\frac{n}{2}.$$

Let

$$m \leq \frac{n}{2}-1.$$

Then from Lemma 1, with m instead of k, it follows

$$D(n-m,m)<D(n-m-1,m+1).$$

Returning to (10) we come to the inequality

$$D(k+1,n-k-1)<D(k,n-k).$$

Hence (9) is proved, because of (1).

As a corollary of Lemma 1 and Lemma 2 we obtain

Theorem Let $n \geq 2$ be a natural number. If:

(a) n is even, then

$$\max_{0 \leq k \leq n} D(n-k,k) = D(\frac{n}{2}, \frac{n}{2});$$

(b) n is odd, then

$$\max_{0 \leq k \leq n} D(n-k,k) = D([\frac{n}{2}]+1, [\frac{n}{2}]) = D([\frac{n}{2}], [\frac{n}{2}]+1).$$

Really, we must note that from Lemma 1 and Lemma 2 it follows immediately that in the case when $n \geq 2$ is even, it is fulfilled:

$$D(n,0) < D(n-1,1) < ... < D(\frac{n}{2}-1, \frac{n}{2}+1) < D(\frac{n}{2}, \frac{n}{2})$$

and

$$D(\frac{n}{2}, \frac{n}{2}) > D(\frac{n}{2}+1, \frac{n}{2}-1) > ... > D(0,n).$$

When $n > 2$ is odd, it is fulfilled:

$$D(n,0) < D(n-1,1) < ... < D(n-[\frac{n}{2}]+1, [\frac{n}{2}]-1) < D(n-[\frac{n}{2}], [\frac{n}{2}])$$

and

$$D(n-[\frac{n}{2}]-1, [\frac{n}{2}]+1) > D(n-[\frac{n}{2}]-2, [\frac{n}{2}]+2) > ... > D(0,n).$$

In particular, we have

$$D(n-[\frac{n}{2}], [\frac{n}{2}]) = D([\frac{n}{2}]+1, [\frac{n}{2}]) = D([\frac{n}{2}], [\frac{n}{2}]+1) = D(n-[\frac{n}{2}]-1, [\frac{n}{2}]+1).$$

For natural number $n \geq 2$ $\max\limits_{0 \leq k \leq n} D(n-k,k)$ is obtained for

$$k = [\frac{n+1}{2}],$$

but when n is odd this maximum is obtained also for

$$k = [\frac{n+1}{2}] - 1.$$

Finally, we must note that when $n \geq 2$ describes the even numbers, then the numbers $D(\frac{n}{2}, \frac{n}{2})$ coincide with the so-called central Delannoy numbers, i.e., the elements of the sequence (see [2]):

$$1, 3, 13, 63, 321, 1683, 8989, ...$$

References

[1] Comtet, L. *Advanced Combinatorics,* D. Reidel Publ. Co. Dordrecht, 1974.

[2] Vassilev M., Atanassov K., On Delanoy numbers, Annuaire de l'Universite de Sofia "St. Kliment Ohridski", Faculte de Mathematiques et Informatique, Livre 1 - Mathematiques, Tome 81, 1987, 153-162.