NNTDM 14 (2008), 2, 1-10

SOME COMBINATORIAL AND RECURRENCE RELATIONS FOR SHAPES IN A TRELLIS

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ABSTRACT

This paper considers the following problems from graph theory: in any section of given size of a trellis or wire-mesh fence, how many squares are there? how many rectangles are there? how many symmetric crosses are there? how many crosses in general ? Certain patterns of arrays of numbers related to various substructures in terms of the numbers of edges and vertices in each case are listed and counted.

1. INTRODUCTION

The planar representation of a trellis (or wire mesh) fence consists of sets of 'crosses' or 'squares' as shown in Figure 1.

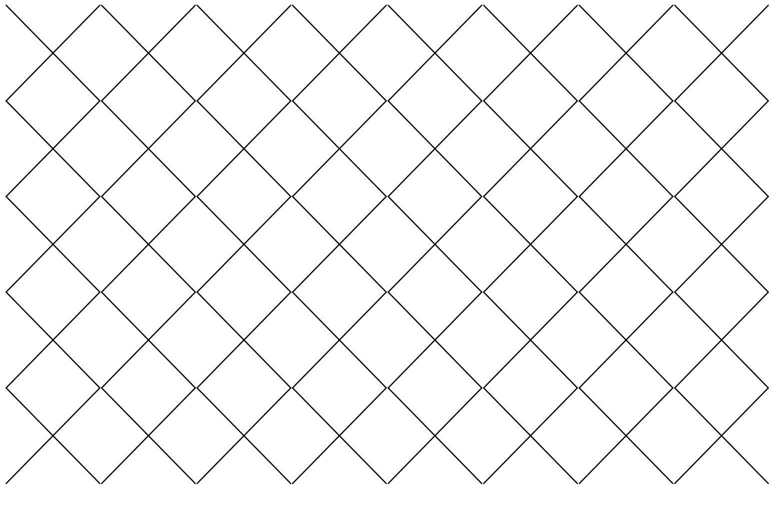

Figure 1: Representation of a section of trellis (wire-mesh)

The first author in 2002 (when aged 10) raised a number of non-trivial questions, namely, in any representation of a trellis fence of a specified size

- how many squares are there?
- how many symmetric crosses are there?
- how many rectangles are there?
- how many crosses, symmetric or asymmetric, are there?

Attempts to solve the problems are probably best illustrated by construction. In general, one would expect the solutions to be functions of the numbers of edges and vertices.

We first define a trellis of a given size and position in the plane as even or odd:

- an even trellis, $f_{n,m}$, is the set of diagonally joined integer lattice points $\{(x, y) : x + y$ is even, $0 \le x \le 2n,\ 0 \le y \le 2m\}$;
- an odd trellis, $g_{n,m}$, is the set of diagonally joined integer lattice points $\{(x, y) : x + y$ is odd, $0 \le x \le 2m,\ 0 \le y \le 2n\}$.

Figures 2 (a),(b),(c),(d) show the cases for $f_{n,m}, n = 1,2,3, m=1,2,3$, respectively, and $g_{3,2}$. Thus in Figure 2, $f_{2,3}$ is the set of single-edged symmetric crosses of 'height' 2 such crosses and 'length' 3 such crosses.

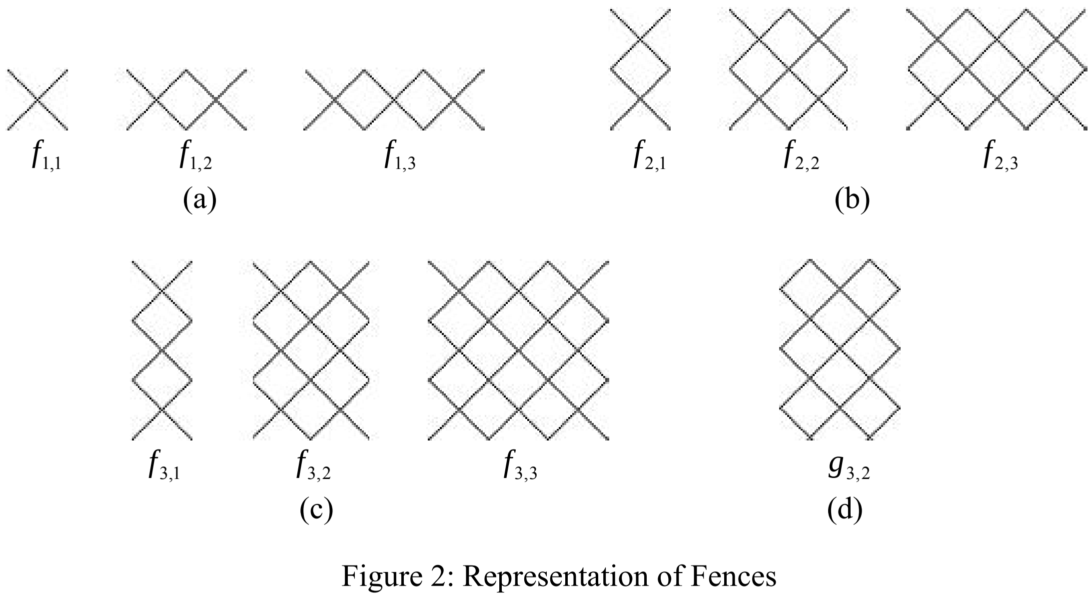

Figure 2: Representation of Fences

2. EDGES AND VERTICES

Let $e(f_{n,m})$ be the number of edges in $f_{n,m}$. Then, since $f_{n,m}$ is constructed by an $n \times m$ lattice of crosses and each cross contributes four edges, it follows that

$$e(f_{n,m}) = 4nm. \qquad (2.1)$$

See the black dots in Figure 3 and the entries in Table 1.

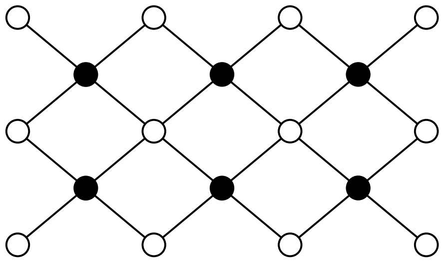

Figure 3: $f_{2,3}$

n↓ m→	1	2	3	4	5
1	4	8	12	16	20
2	8	16	24	32	40
3	12	24	36	48	60
4	16	32	48	64	80

Table 1: $e(f_{n,m})$

Similarly for $e(g_{n,m})$, since $g_{n,m}$ is constructed by $n \times m$ squares and each square consists of four edges

$$e(g_{n,m}) = 4nm\,. \tag{2.2}$$

In addition to having the same recurrence relations we can also see that

$$e(f_{n,m}) = e(g_{n,m})\,.$$

directly from the figures (or a general figure) by noticing that $g_{n,m}$ is obtained from $f_{n,m}$ by removing the first vertical "zig-zag" of $f_{n,m}$ and attaching it to the right hand end, with no change in the number of edges. For edges, the following simple recurrence relations arise from the construction:

$$\begin{aligned} e(f_{n,m}) &= e(f_{n,m-1}) + e(f_{n,1}) \\ &= e(f_{n,m-1}) + 4n \\ &= e(f_{n-1,m}) + e(f_{1,m}) \\ &= e(f_{n-1,m}) + 4m \end{aligned}$$

with $e(f_{1,1}) = 4$. This is readily confirmed in Table 1.

For the corresponding numbers of vertices; $v(f_{n,m})$ and $v(g_{n,m})$, see Table 2. For $v(f_{n,m})$ (see Figure 3), there are *nm* black dots and (*n*+1)(*m*+1) white dots for a total of

$$v(f_{n,m}) = nm + (n+1)(m+1) = 2nm + n + m + 1. \qquad (2.3)$$

n↓ m→	1	2	3	4	5
1	5	8	11	14	17
2	8	13	18	23	28
3	11	18	25	32	39
4	14	23	32	41	50

Table 2: $v(f_{n,m})$

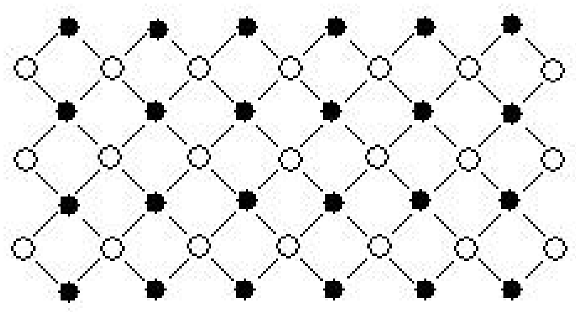

Figure 4: $g_{3,6}$

For $v(g_{n,m})$ (see Figure 4), there are *n*(*m*+1) black dots and *m*(*n*+1) white dots for a total of

$$v(g_{n,m}) = n(m+1) + m(n+1) = 2nm + n + m. \qquad (2.4)$$

For vertices, the following simple recurrence relations again arise from the construction and can be readily confirmed in Table 1.

$$\begin{aligned} v(f_{n,m}) &= v(f_{n,m-1}) + v(f_{n,1}) - (n+1) \\ &= v(f_{n-1,m}) + v(f_{1,m}) - (m+1) \\ &= v(f_{n-1,m}) + 2m + 1 \end{aligned}$$

with $v(f_{1,1}) = 5$. Similarly,

$$\begin{aligned} v(g_{n,m}) &= v(g_{n,m-1}) + v(g_{n,1}) - n \\ &= v(g_{n-1,m}) + v(g_{1,m}) - m \\ &= v(g_{n-1,m}) + 2m + 1 \end{aligned}$$

with $v(g_{1,1}) = 4$. Also we can see that $v(g_{n,m}) = v(f_{n,m}) - 1$ directly from the construction of the figures by noticing that the odd trellis is obtrained from the corresponding even trellis as previously described, with the loss of n+1 vertices and the addition of n vertices.

3. SQUARES AND CROSSES

We now consider the associated squares with edges of any length. A square has an equal number of edges on each side, so that within a trellis a square of side-length k is a set of four points $\{(s,t),(s+k,t-k),(s+k,t+k),(s+2k,t)\}$. More specifically, we consider $s_i(f_{n,m})$, where i is the number of edges in the square (not the graph) and n and m are defined as before. Some examples are displayed in Tables 3 (a),(b),(c),(d).

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8
1	0	1	2	3	4	5	6	7
2	1	4	7	10	13	16	19	22
3	2	7	12	17	22	27	32	37
4	3	10	17	24	31	38	45	5
5	4	13	22	31	40	49	58	67
6	5	16	27	38	49	60	71	82
7	6	19	32	45	58	71	84	97
8	7	22	37	52	67	82	97	112

Table 3(a): $s_1(f_{n,m})$

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	1	2	3	4	5	6	7
3	0	2	5	8	11	14	17	20
4	0	3	8	13	18	23	28	33
5	0	4	11	18	25	32	39	46
6	0	5	14	23	32	41	50	59
7	0	6	17	28	39	50	61	72
8	0	7	20	33	46	59	72	85

Table 3(b): $s_2(f_{n,m})$

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	1	2	3	4	5
4	0	0	1	4	7	10	13	16
5	0	0	2	7	12	17	22	27
6	0	0	3	10	17	24	31	38
7	0	0	4	13	22	31	40	49
8	0	0	5	16	27	38	49	60

Table 3(c): $s_3(f_{n,m})$

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	1	2	3	4	5
5	0	0	0	2	5	8	11	14
6	0	0	0	3	8	13	18	23
7	0	0	0	4	11	18	25	32
8	0	0	0	5	14	23	32	41

Table 3(d): $s_4(f_{n,m})$

Table 3: $s_i(f_{n,m}), i = 1,2,3,4; n,m = 1,2,\ldots,8$

A combinatorial strategy to find a formula for $s_i(f_{n,m})$ is to count the number of possible far-left vertices of squares of side-length i. If i is even, then such vertices will be from an even trellis as in Figure 5, where we are looking at the particular case when i=4. Notice that these dots form a smaller even trellis $f_{n-4,m-4}$.

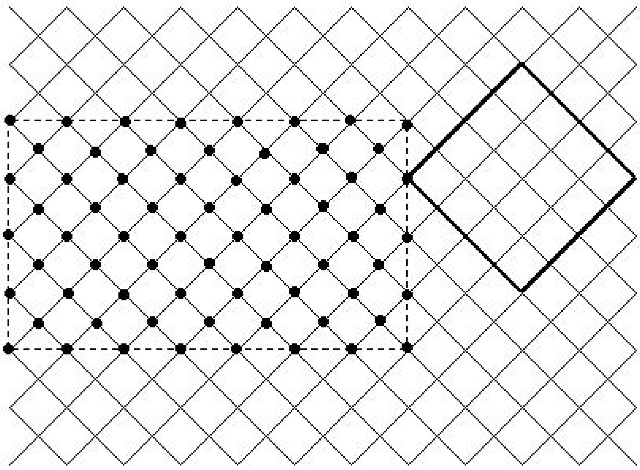

Figure 5: $s_4(f_{8,11})$

While this is only a single case, it suggests that if i is even, then

$$s_i(f_{n,m}) = v(f_{n-i,m-i}) . \tag{3.1}$$

Figure 6 for the odd case is similar, and the dots form a smaller odd trellis $g_{n-5,m-5}$.

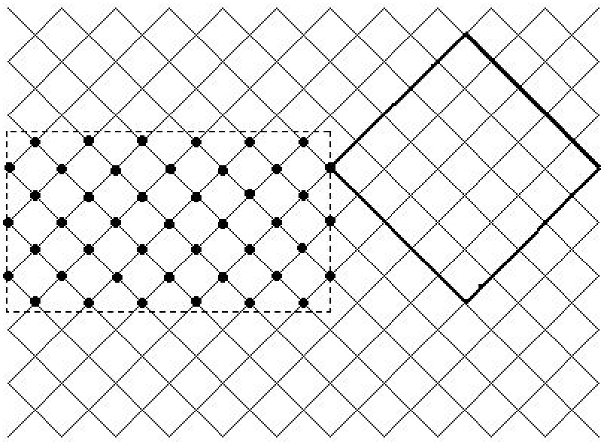

Figure 6: $s_5(f_{8,11})$

This suggests, that if i is odd, then

$$s_i(f_{n,m}) = v(g_{n-i,m-i}) = v(f_{n-i,m-i}) - 1. \tag{3.2}$$

In general, from the construction of the trellis

$$s_i(f_{n,m}) = s_{i-1}(f_{n-1,m-1}) + (-1)^i, \text{ for } n,m \geq i \geq 2. \quad (3.3)$$

Thus,

$$s_i(f_{n,m}) = s_{i-2}(f_{n-2,m-2}), \text{ for } n,m \geq i \geq 3, \quad (3.4)$$

and $s_1(f_{n,m})$ essentially contains all the information required for squares of all sizes (Table 3). A general figure then shows that

$$s_1(f_{n,m}) = s_1(f_{n,m-1}) + s_1(f_{n,2}) - s_1(f_{n,1}), \text{ with } s_1(f_{n,1}) = n-1,$$

so that

$$s_1(f_{n,m}) = 2nm - n - m \quad (3.5)$$

which can be used with (3.3) to yield

$$s_i(f_{n,m}) = \begin{cases} 2nm + (n+m) - 2(n+m+1)i + 2i^2 + d(2,i), & n,m \geq i, \\ 0, & 1 \leq n,m < i, \end{cases} \quad (3.6)$$

in which $d(i,j)$ is the arithmetic divisor function defined by:

$$d(i,j) = \begin{cases} 1, \text{ if } i \mid j, \\ 0, \text{ if } i \nmid j. \end{cases}$$

(3.6) also agrees with (3.2), (3.1) and (2.3).

The above results give

$$s_i(f_{n,m}) = s_i(f_{n,m-1}) + 2(n-i) + 1, \ n,m \geq 1 \geq 1,$$

with

$$s_i(f_{n,i}) = n - i + d(2,i),$$

so, for $m>n$, a closed form can then be obtained for $s(f_{n,m})$ (Table 4) and defined by

$$\begin{aligned} s(f_{n,m}) &= \sum_{i=1}^{n} s_i(f_{n,m}) \\ &= s(f_{n,m-1}) + \sum_{i=1}^{n} \left(2\left(n-1\right)+1\right) \\ &= s(f_{n,m-1}) + n^2 \\ &= s(f_{n,n}) + (m-n)n^2. \end{aligned}$$

Furthermore,

$$\begin{aligned}
s(f_{n,m}) &= \sum_{i=1}^{n} s_i(f_{n,m}) \\
&= \sum_{i=1}^{n} s_1(f_{i,i}) + \lfloor \tfrac{n}{2} \rfloor \qquad \text{using (3.3) and (3.4)} \\
&= \sum_{i=1}^{n} \left(2i^2 - 2i\right) + \lfloor \tfrac{n}{2} \rfloor \qquad \text{using (3.5)} \\
&= \tfrac{2}{3}(n-1)n(n+1) + \lfloor \tfrac{n}{2} \rfloor
\end{aligned}$$

and so

$$s(f_{n,m}) = (m-n)n^2 + \tfrac{2}{3}(n+1)m(n-1) + \lfloor \tfrac{n}{2} \rfloor .$$

Symmetry gives the corresponding result for $n \geq m$, and $s(f_{n,m})$ is in fact symmetric about its main diagonal (which is to be expected as there should be the same number of squares in a $n \times m$ trellis as in a $m \times n$ trellis).

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	2	3	4	5	6	7	8	9	10	11
2		5	9	13	17	21	25	29	33	37	41	45
3			17	26	35	44	53	62	71	80	89	98
4				42	58	74	90	106	122	138	154	170
5					82	107	132	157	182	207	232	257
6						143	179	215	251	287	323	359
7							227	276	325	374	423	472
8								340	404	468	532	596
9									484	565	646	727
10										665	765	865
11											885	1006
12												1150

Table 4: $s(f_{n,m})$, $m \geq n$

For crosses, we have that for every symmetric cross with arms of length i in $f_{n,m}$ there is a corresponding cross of length $i-1$ in $f_{n-1,m-1}$, so that

$$c_i(f_{n,m}) = c_{i-1}(f_{n-1,m-1}),\ n,m \geq i \geq 2 . \qquad (3.7)$$

Thus, as for squares, all information is essentially contained in $c_1(f_{n,m})$. A figure shows

$$
\begin{aligned}
c_1(f_{n,m}) &= c_1(f_{n,m-1}) + c_1(f_{n,1}) + c_1(f_{n-1,1}) \quad \text{(with } c_1(f_{n,1}) = n\text{)} \\
&= c_1(f_{n,m-1}) + 2n - 1 \\
&= c_1(f_{n,1}) + (m-1)(2n-1) \\
&= 2nm - n - m + 1 \qquad \left(= s_1(f_{n,m}) + 1\right)
\end{aligned}
$$

from which, with (3.7), $c_i(f_{n,m})$ is easily obtained.

For crosses of all lengths and $m \geq n$,

$$
\begin{aligned}
c(f_{n,m}) &= \sum_{i=1}^{n} c_i(f_{n,m}) = \sum_{i=1}^{n} c_1(f_{i,m-n+i}) \quad \text{(using (3.7))} \\
&= \sum_{i=1}^{n} \left(2i(m-n+i) - i - (m-n+i) + 1\right) \\
&= \sum_{i=1}^{n} \left(2i^2 + 2i(m-n-1) - (m-n-1)\right) \\
&= \frac{n}{3}\left(1 + 3mn - n^2\right) \quad \left(= s(f_{n,m}) + \lfloor (n+1)/2 \rfloor\right).
\end{aligned}
$$

See Table 6. By symmetry, $c(f_{m,n}) = c(f_{n,m})$.

$n\downarrow$ $m\rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2		6	10	14	18	22	26	30	34	38	42	46
3			19	28	37	46	55	64	73	82	91	100
4				44	60	76	92	108	124	140	156	172
5					85	110	135	160	185	210	235	260
6						146	182	218	254	290	326	362
7							231	2180	329	378	427	476
8								344	408	472	536	600
9									489	570	651	732
10										670	770	870
11											891	1012
12												1156

Table 6: $c(f_{n,m})$, $m \geq n$

4. CONCLUDING COMMENTS

This paper has merely outlined the salient features of the sub-problems in order to bring out the interplay of the combinatorial and number theoretic relations and to open up further related research which could involve

- the inclusion of quadrilaterals in general, as well as other polygons;

- removal of the symmetry conditions for the crosses so that the arms of the crosses may be of unequal length;
- consideration of rectangles as extensions of squares whose adjacent sides (made up of edges) may be unequal;
- enumeration of the squares in a trellis if the restriction of considering only squares with sides parallel to the trellis is removed;
- development of results for crosses of all sizes;
- counting paths of various types joining edge vertices of the trellis;
- determining squares and crosses for the odd trellis and further relationships between the even and odd trellises;
- generalization to higher dimensions;
- investigation of the diagonal sums and partial column and row sums of the various arrays for number sequences: for instance, as the sum of the diagonal entries of $s_1(f_{n,m})$ is $2C_{n-2}^{n+1}$, which is the fourth diagonal of Pascal's triangle, what trellis generalizations might be related to other diagonals?

Combinatorially, spanning trees of various sub-graphs of a trellis (trees of a graph that contain all the vertices of a graph) could also stimulate number theoretic as well as combinatorial results [3].

Educationally, the topics outlined here can be extended to projects at a variety of levels from high school to undergraduate. The problems are extensions of the high school geometry problems of finding the number of triangles in a polygon. The methods too are instructive in that "mathematics is not a deductive science – that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork" [1]. The problems expounded here also lend themselves to the demonstration of notation as a tool of thought [2] as in Section 3 since the symbols themselves are suggestive. This is of importance in the notion of general education *through* mathematics and not merely *in* or *about* mathematics.

Gratitude is expressed to an anonymous referee for detailed constructive advice to improve the paper.

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AMS Classification Numbers 05B35, 51E30.