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INEQUALITIES RELATED TO φ, ψ AND σ-FUNCTIONS (III)

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Here we shall continue the research from [1, 2].

Let us define for the natural number $n \geq 2$:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ... $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers, the following functions (cf., e.g, [3, 4] some of which are used in the present form in others author's papers, too):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1), \text{ and } \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1), \text{ and } \psi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \text{ and } \sigma(1) = 1,$$

$$\underline{cas}(n) = k \text{ and } \underline{cas}(1) = 0,$$

$$d(n) = \prod_{i=1}^{k} (\alpha_i + 1) \text{ and } d(1) = 1,$$

$$\underline{dim}(n) = \sum_{i=1}^{k} \alpha_i \text{ and } \underline{dim}(1) = 0,$$

$$\underline{set}(n) = \{p_1, p_2, ..., p_k\} \text{ and } \underline{set}(1) = \emptyset.$$

Let everywhere below

$$\alpha = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} = 0.607927...,$$

where ζ is Riemann function.

THEOREM 1: For every odd natural number n excluding 3, 5, 9, 15, 27, 45, 75.

$$\alpha.\varphi(n)^2 > \sigma(n).\sqrt{n}. \tag{1}$$

Proof: First, we shall start with three partial cases.

Let $n = 3^a$, where $a \geq 1$ is a natural number. Then

$$\alpha.\varphi(n)^2 - \sigma(n).\sqrt{n} = \alpha.\varphi(3^a)^2 - \sigma(3^a).\sqrt{3^a} = 4\alpha.3^{2(a-1)} - \frac{3^{a+1}-1}{2}.\sqrt{3^a} > 0$$

for $a \geq 4$. It is checked directly that (1) is not valid for $n = 3, 9, 27$.

Let $n = 5^a$, where $a \geq 1$ is a natural number. Then

$$\alpha.\varphi(n)^2 - \sigma(n).\sqrt{n} = \alpha.\varphi(5^a)^2 - \sigma(5^a).\sqrt{5^a} = 16\alpha.5^{2(a-1)} - \frac{5^{a+1}-1}{4}.\sqrt{5^a} > 0$$

for $a \geq 2$. It is checked directly that (1) is not valid for $n = 5$.

Let $n = 3^a5^b$, where $a, b \geq 1$ are natural numbers. Then

$$X \equiv \alpha.\varphi(n)^2 - \sigma(n).\sqrt{n} = \alpha.\varphi(3^a5^b)^2 - \sigma(3^a5^b).\sqrt{3^a5^b}$$

$$= 64\alpha.3^{2(a-1)}5^{2(b-1)} - \frac{(3^{a+1}-1)(5^{b+1}-1)}{8}.\sqrt{3^a5^b}.$$

When $a = 1$, $X > 0$ for $b \geq 3$. It is checked directly that (1) is not valid for $n = 15, 75$.

When $a = 2$, $X > 0$ for $b \geq 2$. It is checked directly that (1) is not valid for $n = 45$.

When $a = 3$, $X > 0$ for every natural number b. Therefore, (1) is valid for every $a \geq 3$ and for every b.

Second, we shall prove Theorem 1 by induction.

Let $n \geq 7$ be a prime number. Then

$$\alpha.\varphi(n)^2 - \sigma(n).\sqrt{n} = \alpha.(n-1)^2 - (n+1).\sqrt{n}$$

$$= \alpha.(n+1)^2 - 4.\alpha.n - (n+1).\sqrt{n} = (n+1).(\alpha.(n+1) - \sqrt{n}) - 4.\alpha.n > 0.$$

Let us assume that (1) is valid for each odd n, different than the above mentioned values, and such that $\underline{dim}(n) = k$ for some natural number $k \geq 1$ and there exists a divisor q of n which is greater than or equal to 7. Let $p \geq 3$ be a given prime number. Therefore, $\underline{dim}(n.p) = k + 1$. For p there are two cases.

First case: p is not a divisor of n.

If $p \geq 7$, then

$$\alpha.\varphi(n.p)^2 - \sigma(n.p).\sqrt{n.p} = \alpha.\varphi(n)^2.(p-1)^2 - \sigma(n).(p+1).\sqrt{n}.\sqrt{p}$$

$$> \sigma(n).\sqrt{n}.((p-1)^2 - (p+1).\sqrt{p})$$

$$= \sigma(n).\sqrt{n}.((p+1).(p+1-\sqrt{p}) - 4p) > 0,$$

because $p + 1 - \sqrt{p} > 4$ for $p \geq 7$.

If $p = 3$ or 5, then we find a prime number $q \geq 7$ that is a divisor of n and construct the number

$$m = \frac{n.p}{q}$$

that is a natural number and $\underline{dim}(m) = k$. Therefore, by assumption

$$\alpha.\varphi(m)^2 > \sigma(m).\sqrt{m}.$$

If $(m, q) = 1$, then we repeat the above check, but for the number mq instead of m. If $(m, q) > 1$ we prove the inequality with mq instead of m by the manner from second case.

Second case: $n = m.p^a$, where m is an odd, $a \geq 1$ and $(m, p) = 1$.

Because

$$\sigma(p^{a+1}) = \frac{p^{a+2}-1}{p-1} = \sigma(p^a).\frac{p^{a+2}-1}{p^{a+1}-1},$$

then

$$\varphi(n.p)^2 - \sigma(n.p).\sqrt{n.p} = \varphi(n)^2.p^2 - \sigma(n).\frac{p^{a+2}-1}{p^{a+1}-1}.\sqrt{n}.\sqrt{p}$$

$$> \sigma(n).\sqrt{n}.(p^2 - \frac{p^{a+2}-1}{p^{a+1}}.\sqrt{p})$$

$$> \sigma(n).\sqrt{n}.(p^2 - (p+1).\sqrt{p}) > 0,$$

since $p^2 - (p+1)\sqrt{3} > 0$ for $p \geq 3$.

Therefore Theorem 1 is proved.

THEOREM 2: For every even natural number $n > 6$

$$3\sqrt{2}.\alpha.\varphi(n)^2 > \sigma(n).\sqrt{n}. \qquad (2)$$

Proof: Let m be an odd number. We shall prove that for every natural number $a \geq 1$

$$3\sqrt{2}.\alpha.\varphi(2^a.m)^2 > \sigma(2^a.m).\sqrt{2^a.m}. \qquad (3)$$

From (1) we obtain for $a = 1$

$$3\sqrt{2}.\alpha.\varphi(2m)^2 = 3\sqrt{2}.\alpha.\varphi(m)^2 > \sigma(2m).\sqrt{2m}.$$

Let us assume that (3) is valid for some natural number $a \geq 1$. Then

$$3\sqrt{2}.\alpha.\varphi(2^{a+1}.m)^2 - \sigma(2^{a+1}.m).\sqrt{2^{a+1}.m}$$

$$= 12\alpha.\sqrt{2}\varphi(2^a.m)^2 - \sigma(2^a.m).\sqrt{2^a.m}.\frac{2^{a+2}-1}{2^{a+1}-1}\sqrt{2}$$

$$= \sigma(2^a.m).\sqrt{2^a.m}.(12\sqrt{2} - \frac{2^{a+2}-1}{2^{a+1}-1}\sqrt{2})$$

$$> 10\sigma(2^a.m).\sqrt{2^{a+1}.m} > 0.$$

Therefore, (3) and, respectively (2) is valid.

COROLLARY 1: For every odd natural number n excluding 3, 5, 9, 15, 45.

$$\alpha.\varphi(n)^2 > \psi(n).\sqrt{n}.$$

COROLLARY 2: For every even natural number $n > 6$

$$3\sqrt{2}.\alpha.\varphi(n)^2 > \psi(n).\sqrt{n}.$$

THEOREM 3: For every $n \geq 5$ and $n \neq 6, 8, 12, 16, 18, 24$:

$$\varphi^3(n) > \sigma^2(n). \tag{4}$$

Proof: Let $n \geq 5$ be a prime number. Then

$$\varphi^3(n) - \sigma^2(n) = (n-1)^3 - (n+1)^2 = n^3 - 4n^2 + n - 2 > 0 \tag{5}$$

for $n \geq 4$.

Let us assume that (4) is valid for some natural number $n \geq 5$ and let $p \geq 5$ be a prime number and $p \notin \underline{set}(n)$. Then

$$\varphi^3(np) - \sigma^2(np) = \varphi^3(n)(p-1)^3 - \sigma^2(n)(p+1)^2 > 0,$$

because (4) is valid by assumption and (5) is proved.

Let $p \in \underline{set}(n)$. Therefore, $n = m.p^a$ for some natural numbers a and m for which $(m,p) = 1$. Then

$$\varphi^3(np) - \sigma^2(np) = \varphi^3(mp^{a+1}) - \sigma^2(mp^{a+1})$$

$$= \varphi^3(m)p^{3a}(p-1) - \sigma^2(m)(\frac{p^{a+2}-1}{p-1})^2$$

$$= \varphi^3(n)p^3 - \sigma^2(n)(\frac{p^{a+2}-1}{p^{a+1}-1})^2$$

$$> \varphi^3(n)p^3 - \sigma^2(n)(p+1)^2 > 0,$$

as above.

Finally, if $n = 2^a$ we obtain

$$\varphi^3(2^a) - \sigma^2(2^a) = (2^{a-1})^3 - (2^{a+1}-1)^2$$

$$= 2^{3a-3} - 2^{2a+2} + 2.2^{a+1} - 1 = 2^{3a-3} - 2^{2a+2} + 2^{a+2} - 1 > 0,$$

for $a \geq 5$. It is checked directly that (4) is not valid for $n = 8, 16$.

If $n = 3^a$ we obtain

$$\varphi^3(3^a) - \sigma^2(3^a) = (2.3^{a-1})^3 - (\frac{3^{a+1}-1}{2})^2$$

$$= 8.3^{3a-2} - \frac{3^{2a+2} - 2.3^{a+1} + 1}{4} > \frac{1}{4}(3^{3a} - 3^{2a+2}) \geq 0,$$

for $a \geq 2$. Therefore, (4) is valid for each $n > 3$ with the present form.

If $n = 2^a 3^b$ we obtain

$$\varphi^3(2^a3^b) - \sigma^2(2^a3^b) = (2^{a-1})^3(2.3^{b-1})^3 - (2^{a+1}-1)^2(\frac{3^{b+1}-1}{2})^2$$

$$= 2^{2a}(2^a 3^{3a-2} - 3^{2b+2}) > 0$$

for:

- $a = 1$ and $b \geq 3$ (but (4) is not valid for $n = 6, 18$),
- $a = 2, 3$ and $b \geq 2$ (but (4) is not valid for $n = 12, 24$),
- $a \geq 4$ and $b \geq 1$.

Therefore, Teorem 3 is proved.

COROLLARY 3: For every $n \geq 5$ and $n \neq 6, 8, 12, 16, 18, 24$:

$$\varphi^3(n) > \psi^2(n).$$

It is clear that for every natural number m there exists a natural number n_0 such that for every natural number $n > n_0$

$$\varphi^{m+1}(n) > \psi^m(n).$$

For example, for $m = 3$ we shall prove

THEOREM 4: For every natural number $n \geq 5$ and $n \neq 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 54, 72, 108, 144, 162, 192, 216, 288, 324, 384, 432, 486, 576$

$$\varphi^4(n) \geq \psi^3(n).$$

Proof: Let $n \geq 5$ be a prime number. Then

$$\varphi^4(n) - \psi^3(n) = (n-1)^4 - (n+1)^3 = n^4 - 5n^3 + 3n^2 - 7n > 0$$

exactly for $n \geq 5$.

Let us assume that the assertion be valid for some natural number $n \geq 5$ and let $p \geq 5$ be a prime number. For p there are two cases.

Case 1: $p \notin \underline{set}(n)$. Then

$$\varphi^4(n.p) - \psi^3(n.p) = \varphi^4(n).(p-1)^4 - \psi^3(n).(p+1)^3 > \psi^3(n).((p-1)^4 - (p+1)^3) > 0.$$

Case 2: $p \in \underline{set}(n)$. Then

$$\varphi^4(n.p) - \psi^3(n.p) = \varphi^4(n).p^4 - \psi^3(n).p^3 > \psi^3(n).p^3.(p-1) > 0.$$

Now, we shall study three special cases.

Let $n = 2^a$. Then

$$\varphi^4(2^a) - \psi^3(2^a) = 2^{4(a-1)} - 3^3.2^{3(a-1)} = 2^{3(a-1)}.(2^{a-1} - 3^3) > 0$$

for $a \geq 6$. It is checked directly that the assertion is not valid for $n = 8, 16, 32$.

Let $n = 3^a$. Then

$$\varphi^4(3^a) - \psi^3(3^a) = 2^4.3^{4(a-1)} - 4^3.3^{3(a-1)} = 2^4.3^{3(a-1)}.(3^{a-1} - 4) > 0$$

for $a \geq 3$. It is checked directly that the assertion is not valid for $n = 9$.

Let $n = 2^a.3^b$. Then

$$X \equiv \varphi^4(2^a.3^b) - \psi^3(2^a.3^b) = 2^4.2^{4(a-1)}.3^{4(b-1)} - 12^3.2^{3(a-1)}.3^{3(b-1)}$$

$$= 2^4.2^{3(a-1)}.3^{3(b-1)}.(2^{a-1}.3^{4(b-1)} - 108).$$

When $a = 1$, $X > 0$ for $b \geq 6$ it is checked directly that the assertion is not valid for $n = 6, 18, 54, 162, 486$.

When $a = 2$, $X > 0$ for $b \geq 5$ it is checked directly that the assertion is not valid for $n = 12, 36, 108, 324$.

When $a = 3$, $X > 0$ for $b \geq 5$ it is checked directly that the assertion is not valid for $n = 24, 72, 216$.

We must note that for $b = 4$, i.e., for $n = 2^3.3^4$ we obtain:

$$\varphi^4(2^3.3^4) = \psi^3(2^3.3^4).$$

When $a = 4$, $X > 0$ for $b \geq 4$ it is checked directly that the assertion is not valid for $n = 48, 144, 432$.

When $a = 5$, $X > 0$ for $b \geq 3$ it is checked directly that the assertion is not valid for $n = 96, 288$.

When $a = 6$, $X > 0$ for $b \geq 3$ it is checked directly that the assertion is not valid for $n = 192, 576$.

When $a = 7$, $X > 0$ for $b \geq 2$ it is checked directly that the assertion is not valid for $n = 384$.

When $a = 8$, $X > 0$ for every $b \geq 1$.

The Theorem is proved.

Let for a fixed prime number p here and below q and r are its successor and its predicessor prime numbers, respectively. Then we can define

$$\rho_+(p) = q,$$

$$\rho_-(p) = r,$$

$$\rho_-(1) = 0,$$

$$\rho_+(1) = 2,$$

$$\rho_-(2) = 1.$$

Obviously, for every prime number p:

$$\rho_-(\rho_+(p)) = p = \rho_+(\rho_-(p))$$

and these equalities can be extended for every natural number n:

$$\rho_-(n) = \prod_{i=1}^{k} \rho_-^{\alpha_i}(p_i),$$

$$\rho_+(n) = \prod_{i=1}^{k} \rho_+^{\alpha_i}(p_i)$$

and

$$\rho_-(\rho_+(n)) = n = \rho_+(\rho_-(n)).$$

Moreover, both functions are multiplicative.

If p is a prime number and n is a natural number, if $p \in \underline{set}(n)$, then $\rho_-(p) \in \underline{set}(\rho_-(n)), \rho_+(p) \in \underline{set}(\rho_+(n))$, and if $p \notin \underline{set}(n)$, then $\rho_-(p) \notin \underline{set}(\rho_-(n)), \rho_+(p) \notin \underline{set}(\rho_+(n))$.

THEOREM 5: For every natural number $n \geq 2$:

(a) $\varphi(\rho_+(n)) \geq n$,

(b) $\sigma(\rho_-(n)) \leq n$,

(c) $\psi(\rho_-(n)) \leq n$.

Proof: (a) Let $n = p$ be a prime number. Then

$$\varphi(\rho_+(n)) = \varphi(q) = q - 1 \geq (n+1) - 2 = n.$$

Let us assume that the inequality is valid for for some natural number n and let p be a prime number. For p there are two cases.

Let $p \notin \underline{set}(n)$. Therefore, $q \notin \underline{set}(\rho_+(n))$ and

$$\varphi(\rho_+(np)) = \varphi(\rho_+(n).q) = \varphi(\rho_+(n)).\varphi(q) \geq n.(q-1) \geq np.$$

Let $p \in \underline{set}(n)$. Therefore, $q \in \underline{set}(\rho_+(n))$ and $n = m.p^a$. Then

$$\varphi(\rho_+(np)) = \varphi(\rho_+(m).q^{a+1}) = \varphi(\rho_+(m)).q^a.(q-1) = \varphi(\rho_+(n)).q \geq n.(p+2) > n.p,$$

i.e. (a) is valid.

(b) and (c) are proved analogously.

THEOREM 6: For every natural number $n \geq 2$, if

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

$$\alpha = \max_i \alpha_i$$

and if $\underline{mindiv}(n) \equiv p_1 < p_2 < \dots < p_k \equiv \underline{maxdiv}(n)$, then

$$(1 + \frac{2}{\underline{maxdiv}(n)})^{\underline{cas}(n)}.n \leq \rho_+(n) \leq (\frac{\rho_+(\underline{maxdiv}(n))}{\underline{mindiv}(n)})^\alpha.n \tag{6}$$

$$(\frac{\rho_-(\underline{mindiv}(n))}{\underline{maxdiv}(n)})^\alpha.n \leq \rho_-(n) \leq (\frac{\rho_+(\underline{maxdiv}(n))}{\underline{mindiv}(n)})^\alpha.n. \tag{7}$$

Proof: Let the natural number $n \geq 2$ be given and let for it $p_1 < p_2 < \dots < p_k$. Then

$$\frac{\rho_+(n)}{n} = \prod_{i=1}^{k}(\frac{\rho_+(p_i)}{p_i})^{\alpha_i} \leq \prod_{i=1}^{k}(\frac{p_{i+1}}{p_i})^{\alpha_i} leq (\prod_{i=1}^{k-1}\frac{p_{i+1}}{p_i})^\alpha.(\frac{\rho_+(p_k)}{p_k})^\alpha = (\frac{\rho_+(\underline{maxdiv}(n))}{\underline{mindiv}(n)})^\alpha.$$

$$\frac{\rho_+(n)}{n} \geq \prod_{i=1}^{k}\frac{\rho_+(p_i)}{p_i} \geq \prod_{i=1}^{k}\frac{p_i+2}{p_i} = \prod_{i=1}^{k}(1+\frac{2}{p_i}) \geq (1+\frac{2}{\underline{maxdiv}(n)})^k$$
$$= (1 + \frac{2}{\underline{maxdiv}(n)})^{\underline{cas}(n)}.$$

Therefore (6) is valid.

$$\frac{\rho_-(n)}{n} = \prod_{i=1}^{k}(\frac{\rho_-(p_i)}{p_i})^{\alpha_i} \leq \prod_{i=1}^{k}(\frac{p_{i+1}}{p_i})^{\alpha_i} \leq (\prod_{i=1}^{k}\frac{p_{i+1}}{p_i})^\alpha = (\frac{\rho_+(\underline{maxdiv}(n))}{\underline{mindiv}(n)})^\alpha.$$

$$\frac{\rho_-(n)}{n} = \prod_{i=1}^{k}(\frac{\rho_-(p_i)}{p_i})^{\alpha_i} \geq (\prod_{i=1}^{k}\frac{\rho_-(p_i)}{p_i})^\alpha \geq (\prod_{i=1}^{k}\frac{p_{i-1}}{p_i})^\alpha = (\frac{\rho_-(p_1)}{p_k})^\alpha$$
$$= (\frac{\rho_-(\underline{mindiv}(n))}{\underline{maxdiv}(n)})^\alpha.$$

Therefore (7) is valid.

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