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A *q*-Series Bernoulli-Euler Partition Formula

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Abstract

This paper utilises a modification of *q*-series to develop some partition formulas in the tradition of Bernoulli-Euler-Rogers-Ramanujan identities. Many of the ideas owe their development to the detailed pioneering work of Leonard Carlitz, with the added acknowledgement to the creative work currently being done by other number theorists working in this fertile area.

1. Introduction

Carlitz has used *q*-series in different ways in numerous papers; for example [2,3,6,7,8,9,15]. Recently, T. Kim and his colleagues have extended some elegant results in both analytic and elementary number theory with such series in a sequence of papers [19-21], and Ernst [18] has provided a current comprehensive history. They are defined basically by

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n), \tag{1.1}$$

with $(q)_0 = 1$. In this brief note, we consider some identities related to the Bernoulli-Euler partition formula

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)} \tag{1.2}$$

which can be used with the two formulae

$$\prod_{n=0}^{\infty}(1-x^n z)^{-1} = \sum_{n=0}^{\infty} z^n/(x)_n \tag{1.3}$$

and

$$\prod_{n=0}^{\infty}(1+x^n z) = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n-1)} z^n/(x)_n \tag{1.4}$$

(the Euler identity), to produce an identity which is not unlike the first Rogers-Ramanujan identity (Andrews [1]}:

$$\sum_{n=0}^{\infty} x^{n^2} /(x)_n = \prod_{n=0}^{\infty}\left(1-x^{n+1}\right)\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(5n+1)}. \tag{1.5}$$

2. Bernoulli-Euler Partition Formula

Carlitz also devoted quite a few papers to the study of partitions [10,11,13,16] and the Rogers-Ramanujan identities [4,5,12,14,17]. While these are not the only papers by Carlitz on these topics, they do contain a representative sample of his techniques and results on these topics.

If we replace z by $-x$ in $\prod_{n=0}^{\infty}\left(1+x^n z\right)$ it becomes $\prod_{n=0}^{\infty}\left(1-x^{n+1}\right)$ which equals $\prod_{n=1}^{\infty}\left(1-x^n\right)$. Thus

$$\prod_{n=1}^{\infty}\left(1-x^n\right)=\sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)} /(x)_n. \tag{2.1}$$

But, from (1.2), we have

$$\begin{aligned}
\prod_{n=1}^{\infty}\left(1-x^n\right) &= \sum_{n=-\infty}(-1)^n x^{\frac{1}{2}n(n+1)} \\
&= \sum_{n=0}^{\infty}(-1)\ \ x^{\frac{1}{2}n(3n+1} + \sum_{n=1}^{\infty}(-1)^n x^{\frac{1}{2}n(3n-1)} \\
&= \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)} - \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}(n+1)(3n+2)} \\
&= \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)}\left(1-x^{2n+1}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)}\left(x^{n^2}-x^{(n+1)^2}\right),
\end{aligned}$$

which agrees with a special case of a formula due to Sylvester [22]. Whence,

$$\sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)} /(x)_n = \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)}\left(x^{n^2}-x^{(n+1)^2}\right), \tag{2.2}$$

which is a formula of the Rogers-Ramanujan type.

3. The First Rogers-Ramanujan Identity

The similarity can be seen more clearly if we carry out the following transformations on the first Rogers-Ramanujan identity.

$$\sum_{n=0}^{\infty} x^{n^2}/(x)_n = \prod_{n=0}^{\infty}\left(1-x^{n+1}\right)^{-1}\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(5n+1)}$$
$$= \sum_{n=0}^{\infty} x^n/(x)_n \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(5n+1)}.$$

Thus,

$$\sum_{n=0}^{\infty} x^{n^2}/(x)_n = \sum_{n=0}^{\infty} x^n/(x)_n \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)}\left(x^{n^2}-x^{(n+1)^2}\right) \qquad (3.1)$$

or

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(5n+1)} = \prod_{n=0}^{\infty}\left(1-x^{n+1}\right)\sum_{n=0}^{\infty} x^{n^2}/(x)_n$$
$$= \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)}/(x)_n \sum_{n=0}^{\infty} x^{n^2}(x)_n,$$

and so

$$\sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(n+1)}/(x)_n \sum_{n=0}^{\infty} x^{n^2}(x)_n = \sum_{n=0}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)}\left(x^{n^2}-x^{(n+1)^2}\right). \qquad (3.2)$$

Identities (3.1) and (3.2) are both forms of the first Rogers-Ramanujan identity.

4. Concluding Comment

Carlitz [17] has also obtained a similar identity by using the Jacobi identity,

$$\sum_{n=-\infty}^{\infty} x^{n^2} \qquad \prod_{n=1}^{\infty}\left(1-x^{2n}\right)\left(1-x^{2n-1}z\right)\left(1-x^{2n-1}z^{-1}\right).$$

References

1. Andrews, G.E. A Polynomial Identity Which Implies the Rogers-Ramanujan Identities. *Scripta Mathematica*. 28 (1970): 297-305.

2. Carlitz, L. q-Bernoulli Numbers and Polynomials. *Duke Mathematical Journal*. 15 (1948): 987-1000.

3. Carlitz, L. Expansions of q-Bernoulli Numbers. *Duke Mathematical Journal*. 25 (1958): 355-364.

4. Carlitz, L. A Note on the Rogers-Ramanujan Identities. *Mathematische Nachrichten.* 17 (1958): 23-26.

5. Carlitz, L. Some Formulas Related to the Rogers-Ramanujan Identities. *Annali di Matematica Pura ed Applicata*. (4) 47 (1959): 243-251.

6. Carlitz L. Some Integral Equations satisfied by the Complete Elliptic Integrals of the First and Second Kind. *Bolletino della Unione Matematica Italiana.* (3) 16 (1961): 264-268.

7. Carlitz, L. Characterization of Certain Sequences of Orthogonal Polynomials. *Portugaliae Mathematica*. 20 (1961): 43-46.

8. Carlitz, L. Generating Functions for Powers of Certain Sequences of Numbers. *Duke Mathematical Journal*. 29 (1962): 521-537.
9. Carlitz, L. A *q*-identity. *Monatshefte für Mathematik.* 67 (1963): 305-310.
10. Carlitz, L. A Problem in Partitions related to the Stirling Numbers. *Bulletin of the American Mathematical Society*. 70 (1964): 275-278.
11. Carlitz, L. Some Partition Problems Related to the Stirling Numbers of the Second Kind. *Acta Arithmetica*. 10 (1965): 409-422.
12. Carlitz, L. Note on Some Continued Fractions of the Rogers-Ramanujan Type. *Duke Mathematical Journal*. 32 (1965): 713-720.
13. Carlitz, L. Rectangular Arrays and Plane Partitions. *Acta Arithmetica.* 13 (1967): 29-47.
14. Carlitz, L. A Note on Products of Sequences. *Bolletino della Unione Matematica Italiana.* (4) 1 (1968): 362-365.
15. Carlitz, L. A Note on the Rogers-Ramanujan Identities. *Duke Mathematical Journal*. 35 (1968): 839-842.
16. Carlitz, L. Fibonacci Representations. II. *The Fibonacci Quarterly.* 8 (1970): 113-134.
17. Carlitz, L. Some Identities in Combinatorial Analysis. *Duke Mathematical Journal*. 38 (1971): 51-56.
18. Ernst, T. Examples of a *q*-umbral Calculus. *Advanced Studies in Contemporary Mathematics.* 16 (2008): 1-22.
19. Kim, T. Some Formulae for the *q*-Bernoulli and Euler Polynomials. *Journal of Mathematical Analysis and Applications*. 273 (2002): 236-242.
20. Kim, T. Analytic Continuation of Multiple *q*-zeta Functions and their Values at Negative Integers. *Russian Journal of Mathematics & Physics.* 11 (2004): 71-76.
21. Kim, T. & S.H. Rim. On Changed *q*-Euler Numbers and Polynomials. *Advanced Studies in Contemporary Mathematics*. 9 (2004): 81-86.
22. Sylvester, J.J. *The Collected Mathematical Papers of James Joseph Sylvester*. Volume IV (1882-1897). New York: Chelsea, 1973, pp.91,93-94.

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