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Arithmetical Function Characterizations and Identities Induced Through Equivalence Relations

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Abstract Let $\mathcal{A}$ denote the set of arithmetic functions and $*$ Dirichlet convolution. The paper presents an alternative approach to the study of arithmetic functions by introducing a homormophism between the subgroup $<\mathcal{U}, *>$ of the group of units in $<\mathcal{A}, *>$ and the quotient ring induced through an equivalence relation. The same notion is extended to the case of unitary convolution.

Keywords: Dirichlet and unitary convolution, functional equation, multiplicative and completely multiplicative functions, equivalence relation, homomorpism.

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1 Introduction

An arithmetic function is a mapping from the positive integers into the field of Complex numbers. We shall denote the set of arithmetic functions by $\mathcal{A}$. Various binary product operations dependent on the divisibility properties of the natural number n may be defined on the set $\mathcal{A}$. Two such well-known products are, the Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}), \tag{1}$$

and the unitary convolution

$$(f \otimes g) = \sum_{\substack{d|n \\ (d,\frac{n}{d})=1}} f(d)g(\frac{n}{d}), \tag{2}$$

where $f, g \in \mathcal{A}$. The latter may also be denoted by, $\sum_{d||n} f(d)g(\frac{n}{d})$. The unitary convolution and related functions have been studied by, amongst others, Cohen [6], [7], Dickson [8], Dirichlet [9], Fekete [10], and Vaidyanathaswamy [17]. Ordinary addition and multiplication on the set $\mathcal{A}$ are defined respectively by: $(f+g)(n) = f(n) + g(n)$ and $(fg)(n) = f(n)g(n)$. It is easily verified that $(\mathcal{A}, +)$ is an abelian group with $\mathbf{0}$, the zero function as the identity. The additive inverse of f is denoted by $-f$. Dirichlet convolution is both commutative and associative. The function

$$\epsilon_0(n) = \begin{Bmatrix} 1; n = 1 \\ 0; n > 1 \end{Bmatrix}$$

is the Dirichlet convolution identity. Further, we note that, if $f(1) \neq 0$, then f has the unique (Dirichlet) inverse denoted by f^{-1} and observe that the set $< \mathcal{A}, *, + >$,with Dirichlet convolution and ordinary addition of functions forms a commutative ring R with identity ϵ_0, Cashwell and Everett [5]. Similarly, it is known, Sivaramakrishnan [16], that $< \mathcal{A}, \otimes, + >$ is a commutative ring with identity and that $< \mathcal{A}, + >$ is an Abelian group.

An arithmetical function is said to be *multiplicative* if,

$$f(1) = 1 \text{ and } f(mn) = f(m)f(n) \text{ whenever } (m,n) = 1,$$

and is called *completely multiplicative* if,

$$f(mn) = f(m)f(n) \text{ for all } m \text{ and } n.$$

It follows therefore, that if f is multiplicative, then $f\left(\prod_{i=1}^{t} p_i^{\alpha_i}\right) = \prod_{i=1}^{t}\left(f(p_i^{\alpha_i})\right)$, and if f is completely multiplicative, then $f\left(\prod_{i=1}^{t} p_i^{\alpha_i}\right) = \prod_{i=1}^{t}\left(f(p_i)\right)^{\alpha_i}$, where $n = \prod_{i=1}^{t} p_i^{\alpha_i} > 1$ and the usual notation is used. Consequently, multiplicative functions are determined by the values $f(p^e)$ and completely multiplicative functions are determined by the values $f(p)$. We shall denote the set of multiplicative functions by $\mathcal{M}$ and the set of completely multiplicative functions by $\mathcal{CM}$.

2 Prime Equivalence Classes

It therefore would be natural to investigate the properties induced on $\mathcal{A}$ by the relations defined by the characterizations above. Thus, on the set $\mathcal{A}$, define the relation $\sim$ as follows;

$$f \sim g \iff f(p) = g(p), \forall \text{ prime } p.$$

It follws that $\sim$ is an equivalence relation. Let $\mathcal{U}$ be the subset of $\mathcal{A}$ such that, for all $f \in \mathcal{U}, f(1) = 1$, so that $\mathcal{M} \subseteq \mathcal{U}$. Further, let $\mathcal{A}/\sim$ denote the quotient set induced by the equivalence relation $\sim$ and $\mathcal{U}/\sim$ the subset of $\mathcal{A}/\sim$ such that, for all $[f] \in \mathcal{U}/\sim, f(1) = 1$, where $[f]$ represents the equivalence class of f.

We now define the following operations on $\mathcal{A}/\sim$ as follows:

1. $[f] + [g] = [f + g]$,

2. $[f] \bullet [g] = [fg]$,

3. $-[f] = [-f]$,

4. $[\mathbf{0}] = \mathbf{0}$, and

5. $[\mathbf{1}] = \mathbf{1}$.

It is easily verified that these operations on $\mathcal{A}/\sim$ are well-defined. We also note that $< \mathcal{A}/\sim, \bullet, + >$ with multiplication and addition as defined above forms a commutative ring.

Definition: Let $n = \prod_{i=1}^{t} p_i^{\alpha_i} > 1$. For each $[f] \in \mathcal{U}/\sim$, define,

$$S_{[f]} : \mathbf{N} \times \mathcal{U}/\sim \longrightarrow \mathbf{C}$$

by

$$S_{[f]}(n) = \sum_{i=1}^{t} \alpha_i f(p_i) \text{ and } S_{[f]}(1) = f(1) = 1,$$

where $f \in [f]$, $\mathbf{N}$ is the set of natural numbers and $\mathbf{C}$ the set of complex numbers.

Next, we define the mappings Φ and Ψ as follows:

$$\Phi :< \mathcal{A}, *, + > \longrightarrow < \mathcal{A}/\sim, \bullet, + >$$

where

$$\Phi(f) = [f],$$

and

$$\Psi :< \mathcal{U}, * > \longrightarrow < \mathcal{U}/\sim, + >,$$

to be the restriction of Φ to $\mathcal{U}$.

The following remark is easily proved and we skip the proof.

Proposition 1 *The function $S_{[f]}$ is uniquely defined, that is, $S_{[f]}(n) = S_{[g]}(n)$, for $n > 1$ and $S_{[f]}(1) = S_{[g]}(1) \iff [f] = [g]$.*

Lemma 2 *The mapping Ψ is a group homormophism.*

Proof

It is easily shown that $< \mathcal{U}, * >$ is a subgroup of the group of units in $< \mathcal{A}, * >$. Further, $\forall f, g \in \mathcal{U}, (f * g)(p) = f(p) + g(p)$. So that,

$$S_{[f*g]}(n) = \sum_{i=1}^{t} \alpha_i (f * g)(p_i) = \sum_{i=1}^{t} \alpha_i f(p_i) + \sum_{i=1}^{t} \alpha_i g(p_i)$$

$$= \sum_{i=1}^{t} \alpha_i (f + g)(p_i) = S_{[f]}(n) + S_{[g]}(n) = S_{[f+g]}(n)$$

$$\text{and } S_{[f*g^{-1}]}(n) = \sum_{i=1}^{t} \alpha_i \left(f * g^{-1}\right)(p_i) = \sum_{i=1}^{t} \alpha_i g^{-1}(p_i) + \sum_{i=1}^{t} \alpha_i f(p_i)$$

$$= \sum_{i=1}^{t} \alpha_i f(p_i) - \sum_{i=1}^{t} \alpha_i g(p_i)$$

$$= S_{[f]}(n) - S_{[g]}(n) = S_{[f-g]}(n), \text{ that is }, [f * g^{-1}] = [f - g]$$

$$\text{with } S_{[f+g]}(1) = S_{[f*g]}(1) = (f * g)(1) = 1.$$

Therefore, since $[h] + [k] = [h + k] = [h * k]$ and $-[h] = [-h] = [h^{-1}]$, it follows that $< \mathcal{U}/ \sim, + >$ is a subgroup of $< \mathcal{A}/ \sim, + >$ and

$$\Psi(f * g) = [f * g] = [f + g] = [f] + [g] = \Psi(f) + \Psi(g).$$

Remark: Let $f_i, g_j \in \mathcal{U}$, where $1 \leq i \leq r$ and , $1 \leq j \leq s$. Then,

$$[f_1 * f_2 * \cdots * f_r] = [f_1 + f_2 + \cdots f_r] = \sum_{i=1}^{r} [f_i],$$

and

$$[f_1 * f_2 * \cdots * f_r * g_1^{-1} * g_2^{-1} * \cdots * g_s^{-1}] = [f_1 + f_2 + \cdots + f_r - g_1^{-1} - g_2^{-1} - \cdots - g_s^{-1}]$$

$$= \sum_{i=1}^{r}[f_i] - \sum_{i=1}^{s}[g_i].$$

This follows by induction from above.

Lemma 3 *Let $m, n > 1$. Then*

1. $S_{[f]}(mn) = S_{[f]}(m) + S_{[f]}(n)$, *and*

2. $S_{[f]}(n^r) = rS_{[f]}(n)$, *where r is a positive integer.*

The proofs follow directly from the definition.

Proposition 4 *Let $f, g \in \mathcal{M}$. If n is square-free, then*

$$[f] = [g] \iff f = g.$$

And as an application, since;

$$S_{[f^{-1}]}(n) = -\sum_{i=1}^{t}\alpha_i f(p_i) = \sum_{i=1}^{t}\alpha_i(\mu f)(p_i) = S_{[\mu f]}(n),$$

that is, $[f^{-1}] = [\mu f]$, it follows that, $f^{-1} = \mu f$, whenever n is square-free.

Examples

Example 1

Let $N(n) = n$ and $u(n) = 1, \forall$ natural numbers n. Then,

$$S_{[\phi]}(n) = \sum_{i=1}^{t}\alpha_i\phi(p_i) = \sum_{i=1}^{t}\alpha_i p_i - \sum_{i=1}^{t}\alpha_i$$

$$= S_{[N]}(n) - S_{[u]}(n) = S_{[N*\mu]}(n),$$

$$\Rightarrow [\phi] = [N * \mu],$$

where ϕ is Euler's totient function, and μ is the Möbius function, with $u^{-1} = \mu$.

Alternatively,

$$S_{[\phi]}(n) = \sum_{i=1}^{t} \alpha_i \phi(p_i) = \sum_{i=1}^{t} \alpha_i p_i - \sum_{i=1}^{t} \alpha_i$$

$$= \sum_{i=1}^{t} \alpha_i (p_i + 1 - 2) = \sum_{i=1}^{t} \alpha_i (p_i + 1) - \sum_{i=1}^{t} \alpha_i \tau(p_i)$$

$$= S_{[\sigma * \tau^{-1}]}(n), \text{so that}, [\phi] = [\sigma * \tau^{-1}] = [\mu * N].$$

Example 2

It is known that $\beta = N * N * \mu$, where $\beta(n) = \sum_{k=1}^{n} (k, n)$ and (k, n) is the gcd of k and n.

Therefore,

$$S_{[N*N*\mu]}(n) = \sum_{i=1}^{t} \alpha_i p_i + \sum_{i=1}^{t} \alpha_i p_i - \sum_{i=1}^{t} \alpha_i$$

$$= \sum_{i=1}^{t} \alpha_i p_i + \sum_{i=1}^{t} \alpha_i (p_i - 1)$$

$$= \sum_{i=1}^{t} \alpha_i N(p_i) + \sum_{i=1}^{t} \alpha_i \phi(p_i)$$

$$= S_{[N*\phi]}(n), \text{that is}, [\beta] = [N * \phi].$$

Alternatively,

$$S_{[N*N]}(n) = \sum_{i=1}^{t} 2\alpha_i p_i = \sum_{i=1}^{t} \alpha_i \tau N(p_i)$$

$$= S_{[\tau N]}(n), \text{that is}, [N * N] = [\tau N] = [\beta * u].$$

Example 3

Now let, $f, g, h \in \mathcal{U}$ and consider,

$$S_{[f(g*h)]}(n) = \sum_{i=1}^{t} \alpha_i (fg)(p_i) + \sum_{i=1}^{t} \alpha_i (fh)(p_i)$$

$$= S_{[fg]}(n) + S_{[fh]}(n) = S_{[fg*fh]}(n), \text{ that is }, [f(g * h)] = [fg * fh].$$

Example 4

Next, we let $f \in \mathcal{CM}$ and consider,

$$S_{[f]}(n) = S_{[f*f*f^{-1}]}(n) = \sum_{i=1}^{t} 2\alpha_i f(p_i) + \sum_{i=1}^{t} \alpha_i f^{-1}(p_i)$$

$$= \sum_{i=1}^{t} \alpha_i (\tau f)(p_i) + \sum_{i=1}^{t} \alpha_i (\mu f)(p_i)$$

$$= S_{[\tau f * \mu f]}(n), \text{that is, } [f] = [\tau f * \mu f] = [f(\tau * \mu)].$$

It is therefore clear, that some of these equivalence classes can give rise to arithmetical identities and or characterizations, but first, we have the following result.

Proposition 5 *Let $f, g \in \mathcal{CM}$. Then $f = g \iff [f] = [g]$.*

Proof

The 'if' condition follows directly from the lemma. Conversely, suppose $[f] = [g], \Rightarrow f(p) = g(p) \forall$ prime p.

$$\Rightarrow f\left(\prod_{i=1}^{t} p_i^{\alpha_i}\right) = \prod_{i=1}^{t} f(p_i^{\alpha_i}) = \prod_{i=1}^{t} f(p_i)^{\alpha_i}$$

$$= \prod_{i=1}^{t} g(p_i)^{\alpha_i} = g\left(\prod_{i=1}^{t} p_i^{\alpha_i}\right), \text{ since } f, g \in \mathcal{CM},$$

that is, $f(n) = g(n) \forall n$. And, hence, the next result.

Corollary 1 *Let $f, g, h \in \mathcal{U}$ and $g * h = u \Rightarrow [f] = [fg * fh]$. Then, $f = fg * fh \iff f \in \mathcal{CM}$.*

Particular cases of the above result would follow from the solution of $g * h = u$. Alternatively, the group homormophism Ψ allows for the solution of the much simpler equation; $[g + h] = [u]$ or $g(p) + h(p) = 1$.

Thus,

1. $\tau(p)+\mu(p) = 1$ giving rise to $\tau*\mu = u$ and hence the result, $f \in \mathcal{CM} \iff f = f(\tau*\mu)$.

   We note that this result is equivalent to the result $f \in \mathcal{CM} \iff f * f = f\tau$, see Carlitz [3].

2. $N(p) - \phi(p) = 1$ giving rise to $N * \phi^{-1} = u \to N = u * \phi$ and hence the result $f \in \mathcal{CM} \iff f = fN * f\phi^{-1}$ or $fN = f * \left(f\phi^{-1}\right) = f * f\phi$, where we use the fact that, $(fg)^{-1} = fg^{-1} \iff f \in \mathcal{CM}$ as proved below. This is a known result, see Sivaramakrishnan[15].

3. $\sigma(p) - N(p) = 1$ giving rise to $\sigma * N^{-1} = u$ and hence the result $f \in \mathcal{CM} \iff f = f\sigma * fN^{-1}$ or $f\sigma = f * fN$.

4. $N_k(p) - J_k(p) = 1$ giving rise to $N_k * J_k^{-1} = u$ and hence the result $f \in \mathcal{CM} \iff f = fN_k * fJ_k^{-1}$ or $fN_k = f * fJ_k$, where J_k is Jordan's totient function and $N_k(n) = n^k$.

   We note that this result generalizes result 2 above.

Corollary 2 *Let $f, g, h \in \mathcal{U}$ and $g * h = \epsilon_0 \Rightarrow [\epsilon_0] = [fg * fh]$. Then $\epsilon_0 = fg * fh \iff f \in \mathcal{CM}$. Alternatively, $(fg)^{-1} = fg^{-1} \iff f \in \mathcal{CM}$, see Apostol [1].*

One well known result, $f \in \mathcal{CM} \iff f^{-1} = \mu f$ derives from the fact that, $u * \mu = \epsilon_0$. Clearly, many more such results could be found from the solution of $g * g^{-1} = \epsilon_0$.

The above corollaries generalize to give the following well known result, see Apostol [1].

Theorem 6 *Let $f \in \mathcal{M}$. Then, $f \in \mathcal{CM} \iff f(g * h) = fg * fh \forall g, h \in \mathcal{A}$.*

Proof

It is clear that if $f \in \mathcal{CM}$ then, $f(g * h) = fg * fh$. And the converse follows by choosing g and h such that $g * h = u$ or $g * h = \epsilon_0$ and using any of the above corrolaries.

If in theorem 6, we choose $h = u$ we obtain the result, $f \in \mathcal{CM} \iff f * fg = fG$ where $f, G \in \mathcal{M}$ and $g = G * u$, see Shonhiwa [14].

Next, we define:

$$F_{[f]}(n) = \sum_{d|n} S_{[f]}(d), \text{ that is } F_{[f]} = u * S_{[f]}.$$

Then,

$$S_{[F_{[f]}]}(n) = S_{[u]}(n) + S_{S_{[f]}}(n) = \sum_{i=1}^{t} \alpha_i u(p_i) + \sum_{i=1}^{t} \alpha_i S_{[f]}(p_i)$$

$$= \sum_{i=1}^{t} \alpha_i \left(u + S_{[f]}\right)(p_i), \text{ that is, } [F_{[f]}] = [u + S_{[f]}].$$

Equivalently, $[F_{[f]}] = [u + \frac{\tau}{2} S_{[f]}]$ which raises the question of whether or not

$$F_{[f]}(n) = \sum_{d|n} S_{[f]}(d) = \left(u + \frac{\tau}{2} S_{[f]}\right)(n)?$$

On letting $n = p^e$, we find that,

$$F_{[f]}(p^e) = \sum_{d|p^e} S_{[f]}(d) = 1 + f(p)\left(1 + 2 + 3 + \cdots + e\right)$$

$$= 1 + \frac{f(p)e(e+1)}{2} = (u + \frac{\tau}{2} S_{[f]})(p^e).$$

And for $n = mk, (m, k) = 1$ we find that,

$$\sum_{d|mk} S_{[f]}(d) = 1 + \sum_{\substack{d|mk \\ d>1}} S_{[f]}(d) = 1 + \sum_{\substack{d_1 d_2 | mk,\, d_1 d_2 > 1 \\ (d_1, d_2) = 1}} S_{[f]}(d_1 d_2)$$

$$= 1 + \sum_{\substack{d_1|m,\, d_2|k \\ d_1 \neq 1}} S_{[f]}(d_1) + \sum_{\substack{d_2|k,\, d_1|m \\ d_2 \neq 1}} S_{[f]}(d_2)$$

$$= 1 + \frac{\tau(k)\tau(m)}{2} S_{[f]}(m) + \frac{\tau(m)\tau(k)}{2} S_{[f]}(k),$$

$$= 1 + \frac{\tau(m)\tau(k)}{2} \left(S_{[f]}(m) + S_{[f]}(k)\right) = 1 + \frac{\tau(mk)}{2} S_{[f]}(mk),$$

where we have used lemma 3, induction on n and the multiplicativity of the function τ. And, hence, the following result.

Theorem 7 $F_{[f]}(n) = \sum_{d|n} S_{[f]}(d) = 1 + \frac{\tau(n)}{2} S_{[f]}(n)$.

In the literature, results close to the above were obtained by, Chawla [2] and LeVan [13].

3 Applications

We now turn our attention to the elegant results obtained in a paper by Carlitz and Subbarao, [4].

Let $f \in \mathcal{U}$ and define for $n = 1, 2, \cdots$

$$\bar{f}(1) = 1, \bar{f}(n) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \sum_{\substack{d_1 d_2 \cdots d_r = n \\ d_i \neq 1, i = 1, 2, \cdots, r}} f(d_1) f(d_2) \cdots f(d_r), n > 1;$$

$$\tilde{f}(1) = 1, \tilde{f}(n) = \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{\substack{d_1 d_2 \cdots d_r = n \\ d_i \neq 1, i = 1, 2, \cdots, r}} f(d_1) f(d_2) \cdots f(d_r), n > 1,$$

where each of the inner summations is over all the sets $(d_1, d_2, \cdots d_r)$ satisfying the given conditions. Further, we let β and γ denote respectively the transformations on $\mathcal{U}$ defined by,

$$\beta(f) = \bar{f} \text{ and } \gamma(f) = \tilde{f}, f \in \mathcal{U}.$$

The authors show that β is a bijection on $\mathcal{U}$ and that its inverse is the mapping γ. They then proceed to use some of their results to solve the functional equation; $f^{(k)} = g, f, g \in \mathcal{U}$ for given g, where $f^{(k)} = f * f * \cdots * f (k \text{ times })$.

First; since Ψ is a group homormophism and $\bar{f}(p) = f(p), \forall$ prime p, it follows that,

$$\Psi\left(\overline{f * g}\right) = [\overline{f * g}] = [f * g]$$

$$= [f+g] = [f] + [g] = [\bar{f}] + [\bar{g}]$$

$$= [\bar{f} + \bar{g}] = \Psi(\bar{f}) + \Psi(\bar{g}),$$

so that,

$$[\overline{f * g}] = [\bar{f} + \bar{g}].$$

Similarly,

$$\Psi(\tilde{f} * \tilde{g}) = [\tilde{f} * \tilde{g}] = [\tilde{f} + \tilde{g}]$$

$$= [\tilde{f}] + [\tilde{g}] = [f] + [g]$$

$$= [f+g] = [\widetilde{f+g}],$$

where we used $\tilde{f}(p) = f(p), \forall$ prime p, and hence,

$$[\tilde{f} * \tilde{g}] = [\widetilde{f+g}].$$

Further, it is known that if $f, g \in \mathcal{U}$ and $f \in \mathcal{CM}$, then $\overline{fg} = f\bar{g}$, [4]. We now prove the following theorem.

Theorem 8 *Let $f, g \in \mathcal{U}$. Then, $f \in \mathcal{CM} \iff \overline{fg} = f\bar{g}$.*

Proof

We skip the 'if' part which is straightfoward using the definition. We now assume that, $\overline{fg} = f\bar{g}, \forall g \in \mathcal{U}$ and show $f \in \mathcal{CM}$.

Choose $g = u$ which is completely multiplicative. Then,

$$\overline{fu} = f\bar{u} = \bar{f}u = \bar{f} \Rightarrow \bar{f}(p^e) = f(p^e)\bar{u}(p^e).$$

But, for $n > 1$,

$$\bar{u}(n) = \left\{ \begin{array}{c} \frac{1}{e}; n = p^e, e = 1, 2, \cdots \\ 0;\ \text{otherwise} \end{array} \right\}, [4].$$

Therefore,

$$f(p^e) = e\bar{f}(p^e) = e\left\{\sum_{j=0}^{e-1}\frac{(-1)^j}{(j+1)}\binom{e-1}{j} f^e(p) - f^e(p) + f(p^e)\right\}$$

$$= e\left\{\frac{f^e(p)}{e} - f^e(p) + f(p^e)\right\}$$

$$\Rightarrow f(p^e) = f^e(p),$$

and, hence, the result. We note that the formula for $\bar{f}(p^e)$ was derived inductively. Haukkanen [12], also proved this result, albeit using a different argument.

Using this result, we reprove (more efficiently) the following result.

Theorem 9 *Let $f \in \mathcal{CM}$. Then $\bar{f}(1) = 1$ and for $n > 1$,*

$$\bar{f}(n) = \left\{\begin{array}{c} \frac{1}{e}(f(p))^e; n = p^e, e = 1, 2, \cdots \\ \\ 0;\ otherwise \end{array}\right\}, [4].$$

Proof

From $\bar{f} = \overline{fu} = f\bar{u}$, it follows that,

$$\bar{f}(n) = f(n)\bar{u}(n) = \left\{\begin{array}{c} \frac{1}{e}(f(p))^e; n = p^e, e = 1, 2, \cdots \\ \\ 0;\ \text{otherwise} \end{array}\right\}.$$

Next define the mapping;

$$\Gamma : < \mathcal{U}/\sim, + > \longrightarrow < \mathcal{U}, * >,$$

by

$$\Gamma([\bar{f}]) = \beta(f).$$

We first obtain the following result.

Theorem 10 *The composition*

$$\Psi \circ \beta : < \mathcal{U}, * > \longrightarrow < \mathcal{U}/\sim, + >$$

is a group homomorphism.

Proof

$$(\Psi \circ \beta)(f * g) = \Psi(\beta(f * g))$$

$$= [\beta(f * g)] = [\bar{f} + \bar{g}] = [\bar{f}] + [\bar{g}]$$

$$= \Psi(\beta(f)) + \Psi(\beta(g)) = (\Psi \circ \beta)(f) + (\Psi \circ \beta)(g).$$

Now, in order to solve the functional equation $f^{(k)} = g$ we shall require the following theorem, by Glöckner, Lucht and Porubský [11].

Theorem 11 *Let $g \in \mathcal{U}$ and k a natural number. The equation $f^{(k)} = g$ has k distinct solutions given by, $f = \omega_i h, i = 1, 2, \cdots, k$, where ω_i are the k-th roots of unity and h is one solution to $f^{(k)} = g$.*

Applying the homomorphism, we note that solving $f^{(k)} = g$ is equivalent to solving $[k\bar{f}] = [\bar{g}]$. But, $\Gamma\left([k\bar{f}]\right) = k\bar{f}$, from which it follows that,

$$k\bar{f} = \Gamma\left([k\bar{f}]\right) = \Gamma([\bar{g}]) = \beta(g)$$

$$\Rightarrow \bar{f} = \frac{1}{k}\bar{g} \Rightarrow f = \widetilde{\frac{1}{k}\bar{g}},$$

and, hence,

$$f = exp(\frac{2\pi i s}{k})\widetilde{\frac{1}{k}\bar{g}}, s = 0, 1, \cdots, k-1.$$

A similar argument can be used for solving the equation, $f^{(k)} = fg$ where $g \in \mathcal{CM}$ and $g(n) \neq k$ for any positive integer n.

4. Prime Power Equivalence Classes: We now turn briefly to the second part of our investigation. Proceeding as before, we define the relation $\sim$ as follows:

$f \sim g \iff f(p^e) = g(p^e), \forall$ prime p and $e \geq 0$. It follows that $\sim$ is an equivalence relation.

Definition: Let $n = \prod_{i=1}^{t} p_i^{\alpha_i} > 1$. For each $[f] \in \mathcal{U}/\sim$, define,

$$S_{[f]} : \mathbf{N} \times \mathcal{U}/\sim \longrightarrow \mathbf{C}$$

by

$$S_{[f]}(n) = \sum_{i=1}^{t} f(p_i^{\alpha_i}) \text{ and } S_{[f]}(1) = f(1) = 1,$$

where $f \in [f]$, $\mathbf{N}$ is the set of natural numbers and $\mathbf{C}$ the set of complex numbers.

We shall redefine the mappings Φ and Ψ as follows:

$$\Phi :< \mathcal{A}, \otimes, + > \longrightarrow < \mathcal{A}/\sim, \bullet, + >$$

where

$$\Phi(f) = [f],$$

and

$$\Psi :< \mathcal{U}, \otimes > \longrightarrow < \mathcal{U}/\sim, + >,$$

to be the restriction of Φ to $\mathcal{U}$. We note that Ψ is a group homomorhism on account of the fact that

$$(f \otimes g)(p^e) = \sum_{d||p^e} f(d) g(\frac{p^e}{d}) = f(p^e) + g(p^e).$$

So that,

$$S_{[f \otimes g]}(n) = \sum_{i=1}^{t} (f \otimes g)(p_i^{\alpha_i})$$

$$= \sum_{i=1}^{t} f(p_i^{\alpha_i}) + \sum_{i=1}^{t} g(p_i^{\alpha_i}) = S_{[f+g]}(n).$$

Also,

$$S_{[f \otimes g^{-1}]}(n) = \sum_{i=1}^{t} f(p_i^{\alpha_i}) + \sum_{i=1}^{t} g^{-1}(p_i^{\alpha_i})$$

$$= S_{[f]}(n) + S_{[g^{-1}]}(n) = S_{[f-g]}(n).$$

The next result is important and easy to prove. We skip the proof.

Theorem 12 *Let $f, g \in \mathcal{M}$. Then, $S_{[f]}(n) = S_{[g]}(n) \iff f = g$.*

Examples

1.

$$S_{[N\otimes u]}(n) = \sum_{i=1}^{t} N(p_i^{\alpha_i}) + \sum_{i=1}^{t} u(p_i^{\alpha_i})$$
$$= \sum_{i=1}^{t}(1 + p_i^{\alpha_i}) = S_{[\sigma^*]}(n),$$

where $\sigma^*(n) = \sum_{d||n} d$. It can be shown that σ^* is multiplicative, so that, $N \otimes u = \sigma^*$.

2.

$$S_{[N\otimes N]}(n) = \sum_{i=1}^{t} N(p_i^{\alpha_i}) + \sum_{i=1}^{t} N(p_i^{\alpha_i}) = \sum_{i=1}^{t} 2p_i^{\alpha_i}$$
$$= \sum_{i=1}^{t} \tau^* N(p_i^{\alpha_i}) = S_{[\tau^* N]}(n),$$

where $\tau^*(p^e) = \sum_{d||p^e} 1 = 2$. That is, $N \otimes N = \tau^* N$ and if n is square-free, then $N \otimes N = N * N = \tau N$.

Next we shall define the unitary analogue of the Möbius function by, $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct primes of n and $\omega(1) = 0$. We may evaluate $\sum_{d||n} \mu^*(d)$ as follows:

$$S_{[u\otimes\mu^*]}(n) = S_{[u]}(n) + S_{[\mu^*]}(n)$$
$$= \sum_{i=1}^{t} u(p_i^{\alpha_i}) + \sum_{i=1}^{t} \mu^*(p_i^{\alpha_i})$$
$$= \sum_{i=1}^{t} 1 + \sum_{i=1}^{t}(-1) = \sum_{i=1}^{t} \epsilon_0(p_i^{\alpha_i}) = S_{[\epsilon_0]}(n),$$

that is, $u \otimes \mu^* = \epsilon_0$. Now, it is known that if $\zeta \otimes \eta = \epsilon_0$, then

$$f = \zeta \otimes g \iff g = \eta \otimes f, \text{ Cohen [7].}$$

Combining this result with the previous one, it follows that,

$$f = u \otimes g \iff g = \mu^* \otimes f.$$

Whilst some of the above results are standard, the homomorphism provides an alternative approach to the study of unitary convolution. Further, the discussion immediately above raises other questions, for instance, do we still have the analogues of the functions β and α on $< \mathcal{U}, \otimes >$? These and other questions raised by the foregoing investigations will be addressed in subsequent researches.

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