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On some Pascal's like triangles. Part 1

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In a series of papers we shall discuss new types of Pascal's like triangles. Triangles from the present form, but not with the present sense, are described in different publications, e.g. [1, 2, 3], but at least the author had not found a research with similar idea.

In the first part of our research we shall study properties of "standard" sequences, while in the next parts the objects of our interest will be more non-standard sequences.

First, let us construct the following infinite triangle:

						1						
					1	**2**	1					
				1	2	**4**	2	1				
			1	2	4	**8**	4	2	1			
		1	2	4	8	**16**	8	4	2	1		
	1	2	4	8	16	**32**	16	8	4	2	1	
1	2	4	8	16	32	**64**	32	16	8	4	2	1
				.		.		.				

We see that the terms in the middle column are exactly the consequential powers of number 2. Now, we can put these powers on the two boundary diagonals, the left and the right, and we will obtain infinite triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 2 & \mathbf{3} & 2 & & & & & \\
 & & & & 4 & 6 & \mathbf{9} & 6 & 4 & & & & \\
 & & & 8 & 12 & 18 & \mathbf{27} & 18 & 12 & 8 & & & \\
 & & 16 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 16 & & \\
 & 32 & 48 & 72 & 108 & 162 & \mathbf{243} & 162 & 108 & 72 & 48 & 32 & \\
64 & 96 & 144 & 216 & 324 & 486 & \mathbf{729} & 486 & 324 & 216 & 144 & 96 & 64 \\
 & & & & . & & . & & . & & & &
\end{array}$$

which terms that stay in the middle column are exactly the consequential powers of number 3.

Let the elements of the above infinite triangles be

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & . & & . & & . &
\end{array}$$

where $a_{i,j}$ are arbitrary real (complex) numbers and for every natural number i and
1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i,j-1} + a_{i-1,j-1};$$

2. for every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1} + a_{i-1,j-1}.$$

We can prove, e.g., by induction

Lemma 1: For every natural number n:

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & n & \mathbf{n+1} & n & & \\
 & n^2 & n^2+n & (\mathbf{n+1})^{\mathbf{2}} & n^2+n & n & \\
n^3 & n^3+n^2 & n^3+2n^2+n & (\mathbf{n+1})^{\mathbf{3}} & n^3+2n^2+n & n^3+n^2 & n^3 \\
 & . & & . & & . &
\end{array}$$

Let the infinite sequence $\{c_i\}_{i \leq 1}$ of arbitrary real (complex) numbers be given and let for every natural number i:

$$a_{i,1} = a_{i,2i-1} = c_i.$$

Then by induction we can prove

Lemma 2: For every natural number $j : 1 \leq j \leq i$:

$$a_{i,j} = a_{i,2i-j} = \sum_{k=0}^{i-j} \left(\begin{array}{c} i-j \\ k \end{array} \right) c_{j-k}.$$

On the other hand, if for every natural number i:

$$a_{i,i} = c_i,$$

then we can prove that

$$a_{i,1} = a_{i,2i-1} = \sum_{j=1}^{i} (-1)^{j-1} \left(\begin{array}{c} i \\ j \end{array} \right) c_j.$$

When we like to obtain in the middle column the consequential natural numbers, the elements of the left and right diagonals must be 1,1,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 1 & \mathbf{2} & 1 & & & & \\
 & & & 0 & 1 & \mathbf{3} & 1 & 0 & & & \\
 & & 0 & 0 & 1 & \mathbf{4} & 1 & 0 & 0 & & \\
 & 0 & 0 & 0 & 1 & \mathbf{5} & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 1 & \mathbf{6} & 1 & 0 & 0 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

When we like to obtain in the middle column the consequential squares of the natural numbers, the elements of the left and right diagonals must be 1,3,2,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 3 & \mathbf{4} & 3 & & & & \\
 & & & 2 & 5 & \mathbf{9} & 5 & 2 & & & \\
 & & 0 & 2 & 7 & \mathbf{16} & 7 & 2 & 0 & & \\
 & 0 & 0 & 2 & 9 & \mathbf{25} & 9 & 2 & 0 & 0 & \\
0 & 0 & 0 & 2 & 11 & \mathbf{36} & 11 & 2 & 0 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

When we like to obtain in the middle column the consequential cubes of the natural numbers, the elements of the left and right diagonals must be 1,7,12,6,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 7 & \mathbf{8} & 7 & & & & \\
 & & & 12 & 19 & \mathbf{27} & 19 & 12 & & & \\
 & & 6 & 18 & 37 & \mathbf{64} & 37 & 18 & 6 & & \\
 & 0 & 6 & 24 & 61 & \mathbf{125} & 61 & 24 & 6 & 0 & \\
0 & 0 & 6 & 30 & 91 & \mathbf{216} & 91 & 30 & 6 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

When we like to obtain in the middle column the consequential fourth powers of the natural numbers, the elements of the left and right diagonals must be 1,15,50,64,24,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 15 & \mathbf{16} & 15 & & & & \\
 & & & 50 & 65 & \mathbf{81} & 65 & 50 & & & \\
 & & 60 & 110 & 175 & \mathbf{256} & 175 & 110 & 60 & & \\
 & 24 & 84 & 194 & 369 & \mathbf{625} & 369 & 194 & 84 & 24 & \\
0 & 24 & 108 & 302 & 671 & \mathbf{1296} & 671 & 302 & 108 & 24 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

When we like to obtain in the middle column the consequential fifth powers of the natural numbers, the elements of the left and right diagonals must be 1,31,180,390,360,120,0,0,0,...

If we like to obtain in the middle column the n-th powers of the natural numbers, the elements of the left and right diagonals must be

$$1, 2^n - 1, ..., \frac{(n+1)!}{2}, n!, 0, 0, 0, ...$$

Let a, b be fixed real (complex) numbers. We can construct the infinite triangle

$$\begin{array}{ccccccccc}
 & & & & \mathbf{a} & & & & \\
 & & & b & \mathbf{a+b} & b & & & \\
 & & a & a+b & \mathbf{2a+2b} & a+b & a & & \\
 & b & a+b & 2a+2b & \mathbf{4a+4b} & 2a+2b & a+b & b & \\
a & a+b & 2a+2b & 4a+4b & \mathbf{8a+8b} & 4a+4b & 2a+2b & a+b & a \\
 & & . & & . & & . & &
\end{array}$$

When $b = -a$ the above triangle obtains the form

$$\begin{array}{ccccccc}
 & & & \mathbf{a} & & & \\
 & & -a & \mathbf{0} & -a & & \\
 & a & 0 & \mathbf{0} & 0 & a & \\
-a & 0 & 0 & \mathbf{0} & 0 & 0 & -a \\
 & . & & . & & . &
\end{array}$$

Let a, b, s be fixed real (complex) numbers. We can construct the infinite triangle below, but we show only its left part (the right part is symmetric to it).

$$\begin{array}{cccccccc}
 & & & & & \mathbf{a} & & \\
 & & & & b & \mathbf{a+b} & b & \\
 & & & c & b+c & \mathbf{a+2b+c} & b+c & ... \\
 & & a & a+c & a+b+2c & \mathbf{2a+3b+3c} & a+b+2c & ... \\
 & b & a+b & 2a+b+c & 3a+2b+3c & \mathbf{5a+5b+6c} & 3a+2b+3c & ... \\
c & b+c & a+2b+c & 3a+3b+2c & 6a+5b+5c & \mathbf{11a+10b+11c} & 6a+5b+5c & ... \\
 & & & . & & . & & .
\end{array}$$

When $a=1, b=2, c=3$ this triangle has the form

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{3} & 2 & & & & \\
 & & & 3 & 5 & \mathbf{8} & 5 & 3 & & & \\
 & & 1 & 4 & 9 & \mathbf{17} & 9 & 4 & 1 & & \\
 & 2 & 3 & 7 & 16 & \mathbf{33} & 16 & 7 & 3 & 2 & \\
3 & 5 & 8 & 15 & 31 & \mathbf{64} & 31 & 15 & 8 & 5 & 3 \\
 & & & . & & . & & . & & &
\end{array}
$$

and the following interesting property:

$$a_{3i,3i} = 2^{3i},$$

where $a_{i,j}$ denotes the element of the triangle that lies in the i-th row and in the j-th column.

For the above a, b, c we can construct another infinite triangle (of the preceeding type), too:

$$
\begin{array}{cccccccc}
 & & & & & \mathbf{a} & & \\
 & & & & b & \mathbf{a+b} & b & \\
 & & & c & b+c & \mathbf{a+2b+c} & b+c & ... \\
 & & b & b+c & 2b+2c & \mathbf{a+4b+3c} & 2b+2c & .... \\
 & a & a+b & a+2b+c & a+4b+3c & \mathbf{2a+8b+6c} & a+4b+3c & ... \\
b & a+b & 2a+2b & 3a+4b+c & 4a+8b+4c & \mathbf{6a+16b+10c} & 4a+8b+4c & ... \\
 & & & . & & . & & .
\end{array}
$$

When $a=1, b=2, c=3$ this triangle has the form

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & \mathbf{1} & & \\
 & & & & & & & & & 2 & \mathbf{3} & 2 & \\
 & & & & & & & & 3 & 5 & \mathbf{8} & 5 & ... \\
 & & & & & & & 2 & 5 & 10 & \mathbf{18} & 10 & .... \\
 & & & & & & 1 & 3 & 8 & 18 & \mathbf{36} & 18 & ... \\
 & & & & & 2 & 3 & 6 & 14 & 32 & \mathbf{68} & 32 & ... \\
 & & & & 3 & 5 & 8 & 14 & 28 & 60 & \mathbf{128} & 60 & ... \\
 & & & 2 & 5 & 10 & 18 & 32 & 60 & 128 & \mathbf{248} & 128 & ... \\
 & & 1 & 3 & 8 & 18 & 36 & 68 & 128 & 248 & \mathbf{496} & 248 & ... \\
 & 2 & 3 & 6 & 14 & 32 & 68 & 136 & 264 & 512 & \mathbf{1008} & 512 & ... \\
3 & 5 & 8 & 14 & 28 & 60 & 128 & 264 & 528 & 1040 & \mathbf{2048} & 1040 & ... \\
 & & & & & & . & & . & & . & &
\end{array}
$$

and the following property:

$$a_{4i-1,4i-1} = 2^{4i-1},$$

for $i = 1, 2, 3, ...$

Finally, we construct the infinite triangle of the form

$$\begin{array}{ccccccccccc} & & & & & \mathbf{1} & & & & & \\ & & & & 2 & \mathbf{3} & 2 & & & & \\ & & & 3 & 5 & \mathbf{8} & 5 & 3 & & & \\ & & 4 & 7 & 12 & \mathbf{20} & 12 & 7 & 4 & & \\ & 5 & 9 & 16 & 28 & \mathbf{48} & 28 & 16 & 9 & 5 & \\ 6 & 11 & 20 & 36 & 64 & \mathbf{112} & 64 & 36 & 20 & 11 & 6 \\ & & & . & & . & & . & & & \end{array}$$

with the property:

$$a_{i+1,i+1} = 2a_{i,i} + 2^{i-1}$$

for $i \geq 1$.

References

[1] Bondarenko, B., Generalized Pascal's Triangles and Pyramids - Their Fracals, Graphs and Applications, Tashkent, Fan, 1990 (in Russian).

[2] Goldwasser, J., W. Klostermeyer, M. Mays, G. Trapp, The density of ones in Pascal's rhombus. Discrete mathematics, Vol. 204, 1999, 231-236.

[3] Leyendekkers, J., A. Shannon, J. Rybak. Pattern recognition: Modular Rings & Integer Structure. RafflesKvB Monograph No. 9, North Sydney, 2007.