NNTDM 13 (2007), 1, 16-24

ODD POWERS AS SUMS OF SQUARES

J. V. Leyendekkers
The University of Sydney, 2006, Australia

A. G. Shannon
Warrane College, The University of New South Wales, Kensington, 1465, &
Raffles KvB Institute Pty Ltd, North Sydney, NSW 2060, Australia

Abstract

Odd integers raised to an odd power n can equal a sum of squares only if the integers are in Class $\bar{1}_4$ of the Modular Ring Z_4. The primes raised to an odd power are unique in that they only have (n - 1) square couples. These couples depend on the single couple (a_1, b_1) for p, and on x, a function of p and (a_1, b_1). A general equation is given for predicting the square couples of p^n and odd-powered composites. Even integers raised to an odd power have no primitive solutions for such square couples, because the sum of two odd squares falls in Class $\bar{2}_4$ where there are no powers at all.

1. Introduction

The Modular Ring Z_4 [2,4] (see Table 1) has two classes for odd integers, m. These are $\bar{1}_4$ where $m = 4r_1 + 1$ and $\bar{3}_4$ where $m = 4r_3 + 3$. Integers in $\bar{1}_4$ can equal a sum of squares whereas integers in $\bar{3}_4$ cannot. Fermat showed this to be true many centuries ago. The result follows simply from integer structure. Squares only fall in Class $\bar{1}_4$ when odd, or in $\bar{0}_4$ when even. Therefore, for a sum of squares of opposite parity, we have [2]:

$$\bar{1}_4 + \bar{0}_4 = \bar{1}_4. \tag{1.1}$$

Since all odd powers of odd integers, m, fall in the same class as m, then for odd n, m^n may equal a sum of squares only when $m \in \bar{1}_4$. Thus in this paper we look at the structure of odd powers as a sum of squares.

Row↓ Class→	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
3	12	13	14	15
4	16	17	18	19

Table 1: Rows of the Modular ring Z_4

2. Cubes as a Sum of Squares

A series of integers in $\bar{1}_4$, m_j, may be given by

$$m_j = a^2 + b^2 \\ = (2i+1)^2 + 4j^2, \qquad (2.1)$$

with i constant, and j=1,2,3,4,.. .
Thus,

$$a^2 + b^2 = 4(i^2 + j^2 + i) + 1, \qquad (2.2)$$

and the row of m_j in $\bar{1}_4$ is $(i^2 + j^2 + i)$. For example, with i = 3, the row will be $(12 + j^2)$ and the series is

53,65,85,113,149,193,… ,

or when i = 17, the series is

1229,1241,1261,1289,1325,1369,… .

When i = 2, we have the recurrence relations

$$r_k = r_{k-1} + (2k+1),$$

and

$$m_k = m_{k-1} + 4(2k+1).$$

Such series can be formed for any value of i, but not all integers in $\bar{1}_4$ equal a sum of squares [2]; for instance, 9,21,49,69,93,…,3741,… . In general too, when the factors fall in Class $\bar{3}_4$, the integer does not equal a sum of squares. Of course, the number of factors in $\bar{3}_4$ will be even, since $\bar{3}_4 \times \bar{3}_4 = \bar{1}_4$ but $\bar{3}_4 \times \bar{3}_4 \times \bar{3}_4 = \bar{3}_4$. Primes have only one set of (a_1, b_1) (no common factors), whereas the number of (a_1, b_1) couples for composites depends on the number of factors and the class structure of the factors. If a composite has only one (a_1, b_1) couple then the couple has a common factor. The prime (a_{13}, b_{13}) or (A,B) structure is unique for cubic systems as well (cf. [5]).

<u>Primes</u>

The sum-of-squares characteristics of primes distinguish them from other integers. For primes in Class $\bar{1}_4$ there is only one (a_1, b_1) which does not have a common factor. On the other hand, composites that equal a sum of squares have multiple (a_1, b_1) couples according to the number of factors. Occasionally there is only one couple, in which case it has a common factor [2].

For $p^3 \in \bar{1}_4$, there are two couples which arise as follows. Let

$$p^3 = (k_{13}a_1)^2 + (k_{23}b_1)^2 \\ = A^2 + B^2, \qquad (2.1)$$

in which (a_1, b_1) is the couple of p. for example, the (a_1, b_1) couples for the primes 5,17,73 are (1,2),(1,4) and (3,8) respectively. It is further observed that

$$k_{13} \pm k_{23} = \pm 2p, \tag{2.2}$$

from which two forms arise, as follows, depending on the row of the prime in the $\bar{1}_4$ column of Table 1.

Type 1: Here the rows are odd but $3 \nmid r_1$.

$$k_{13} - k_{23} = -2p. \tag{2.3}$$

Substituting (2.3) into (2.1) we get

$$k_{13} = -2b_1^2 \pm \left(4b_1^4 - p\left(4b_1^2 - p\right)\right)^{\frac{1}{2}}. \tag{2.4}$$

For instance, with $p = 29$, $b_1 = 2$ so that

$$k_{13} = -8 \pm 21 = 13, -29$$

and

$$k_{23} = k_{13} + 2p = 13 + 58 = 71,$$

or

$$k_{23} = -29 + 58 = p.$$

Thus,

$$p^3 = (-1)^2 (a_1 p)^2 + (b_1 p)^2 \tag{2.5}$$

or

$$p^3 = (a_1 k_{13})^2 + (b_1 k_{23})^2 \tag{2.6}$$

in which the *k*s are obtained from (2.3) and (2.4).

Type 2: Here the rows are even unless $3 \mid r_1$.

$$k_{13} - k_{23} = 2p, \tag{2.7}$$

or

$$k_{13} + k_{23} = 2p. \tag{2.8}$$

Both (2.7) and (2.8) when substituted into (2.1) yield

$$k_{13} = 2b_1^2 \pm \left(4b_1^4 - p\left(4b_1^2 - p\right)\right)^{\frac{1}{2}}. \tag{2.9}$$

For example, when $p = 17$, $r_1 = 4$, $b_1 = 4$, so that

$$k_{13} = 32 \pm 15 = 47, 17$$

and

$$k_{23} = 47 - 34 = 13$$

or

$$k_{23} = 34 - 17 = p.$$

Thus, as with Type 1,

$$p^3 = (a_1 p)^2 + (b_1 p)^2 \tag{2.10}$$

or

$$p^3 = (a_1 k_{13})^2 + (b_1 k_{23})^2 \tag{2.11}$$

in which the ks are obtained from (2.3) and (2.4) as before. An example of Equation (2.8) is when $p = 109$, $r_1 = 27$, $b_1 = 10$, so that $k_{13} = 200 \pm 91 = 291{,}109$.

Table 2 provides some examples of (a_1, b_1) and (A,B) couples for various primes.

p	**row**	***a*₁**	***b*₁**	***k*₁₃**	***k*₂₃**	***A***	***B***
13	3	3	2	3 13	23 13	9 39	46 26
17	4	1	4	47 17	13 17	47 17	52 68
29	7	5	2	13 29	71 29	65 145	142 58
37	9	1	6	107 37	33 37	107 37	198 222
41	10	5	4	23 41	59 41	115 205	236 164
53	13	7	2	37 53	143 53	259 371	286 106
61	15	5	6	83 61	39 61	415 305	234 366
73	18	3	8	183 73	37 73	519 219	296 584
109	27	3	10	291 109	73 109	873 327	730 1090

Table 2: Examples of (a_1, b_1) and (A,B), $n = 3$

<u>Composites</u>

The composite integers, N, not surprisingly, have characteristics which are quite distinct from those of prime integers. As noted above, the number of factors of N and the class of each will govern the number of (a_1, b_1) couples and hence the number of (A,B) couples. For example, with $N = 133 = 7 \times 19$ there will be no (a_1, b_1) couples since $7, 19 \in \bar{3}_4$. However, $N \in \bar{3}_4 \times \bar{3}_4 = \bar{1}_4$.

For N which have (a_1, b_1) couples, N^3 will usually have twice, and sometimes three times, the number of (a_1, b_1) couples when forming (A,B) couples. The basic equations are

$$k_{13} = xb_1^2 \pm \left(x^2 b_1^4 - p\left(x^2 b_1^2 - p\right)\right)^{\frac{1}{2}} \tag{2.12}$$

and

$$k_{13} \pm k_{23} = \pm xp. \tag{2.13}$$

As in the case of the primes, the row of N determines the $\pm$ character of (2.13) and $x = 2$ for all N but may also have larger values for some N. The row classifications are also the same as for the primes. Examples of various (A,B) values for a range of N are presented in Table 3. The N values which have $x \geq 2$ are in rows which are even squares. This increases the combinatorial potential of the individual terms of the expanded $(4r_1+1)^3$ in relation to square functions.

m	row	factors	M^3	*a*	*b*	k_1	k_2	*A*	*B*	*x*
25	6	5x5	15625	3	4	25	25	75	100	2
						39	11	117	44	2
45	11	5x3x3	91125	3	6	45	45	135	270	2
						99	-9	-297	-54	2
65	16	5x13	274625	1	8	65	65	65	520	2
						191	61	191	488	2
						415	-40	415	-320	7
						481	26	481	208	7
				7	4	65	65	455	260	2
						-1	131	-7	524	2
85	21	5x17	614125	9	2	-69	239	-621	478	2
						85	85	765	170	2
				7	6	85	85	595	510	2
						59	111	413	666	2
								755	210	-
117	29	3x3x13	1601613	9	6	117	117	1053	702	2
						27	207	243	1242	2
125	31	5x5x5	1953125	11	2	125	125	1375	250	2
						109	359	1199	718	2
				5	10	125	125	625	1250	2
						-275	-25	-1375	-250	2
145	36	5x29	3048625	1	12	145	145	145	1740	2
						431	-141	431	-1692	2
						1745	-5	1745	-60	12
				9	8	145	145	1305	1160	2
						111	179	999	1432	2
221	55	13x17	10793861	5	14	563	-121	2815	-1694	2
						221	221	1105	3094	2
				11	10	179	263	1969	2630	2
						221	221	2431	2210	2
325	81	5x5x13	34328125	1	18	971	-321	971	-5778	2
						325	325	325	5850	2
				15	10	75	575	1125	5750	2
						325	325	4875	3250	2
				17	6	-181	831	-3077	4986	2
						325	325	5525	1950	2

Table 3: Examples of various (A,B) and x values

3. p^5 as a Sum of Squares

When n = 5, the same form of equation applies:

$$p^5 = (a_1 k_{15})^2 + (b_1 k_{25})^2 \tag{3.1}$$

with

$$k_{15} = x b_1^2 \pm \left(x^2 b_1^2 (b_1^2 - p) + p^4\right)^{\frac{1}{2}}. \tag{3.2}$$

For n = 5, one solution if x is always x = 2p, and $k_{15} + k_{25} = xp$. From this value of x, one (k_{15}, k_{25}) couple is (p^2, p^2), while the other always has one negative k, that is (k_{15}, k_{25})= (s,-t) or (-s,t) and for the other solution of x one has (k_{15}, k_{25})= (-s,t) or (s,-t). Hence, since

$$k_{15} + k_{25} = xp, \tag{3.3}$$

x can be found (and note the simpler form for x compared with n = 3). For example, for p = 13, we have s = -39, t = 299, so that x =20, and substitution in Equations (3.2) and (3.3) yields the other (k_{15}, k_{25}). Thus, each square component can be found. The number of (k_{15}, k_{25}) couples for primes is (n-1), and so there are 4 when n = 5. Furthermore, those (k_{15}, k_{25}) that are a function of p follow $p^{\frac{1}{2}(n-1)}$, which for n = 5 becomes p^2. Examples are provided in Table 4. The analysis for composites follows along lines similar to those given for n = 3.

p	**row**	a_1	b_1	k_{15}	k_{25}	**x**
5	1	1	2	55 25	-5 25	10
				55 41	5 19	12
13	3	3	2	39 169	299 169	26
				-39 199	299 61	20
17	4	1	4	799 289	-221 289	34
				799 1121	221 -101	60
29	7	5	2	-377 841	2059 841	58
				377 295	2059 2141	84
37	9	1	6	3959 1369	-1221 1369	74
				6121 3959	-941 1221	140

Table 4: Equation (3.3)

4. General solutions *n* > 3

The fundamental equation is

$$p^n = (a_1 k_{1n})^2 + (b_1 k_{2n})^2 \tag{4.1}$$

with

$$k_{1n} = x b_1^2 \pm \left(x^2 b_1^2 \left(b_1^2 - p\right) + p^{n-1}\right)^{\frac{1}{2}}. \tag{4.2}$$

This can be related to (2.4) and (2.9). There are (*n*-1) values of (k_{1n}, k_{2n}). One set always has

$$k_{1n} = k_{2n} = p^{\frac{1}{2}(n-1)} \tag{4.3}$$

so that

$$x = 2p^{\frac{1}{2}(n-3)}. \tag{4.4}$$

The other (k_{1n}, k_{2n}) corresponding to this *x* has the form (-*s*,*t*) or (*s*,-*t*). The next value of *x* yields one (k_{1n}, k_{2n}) set equal to (+*s*,*t*) or (*s*,+*t*) which gives *x* from

$$k_{1n} + k_{2n} = xp \tag{4.5}$$

and hence the associated (k_{1n}, k_{2n}) pair. This pair indicates one of the next (k_{1n}, k_{2n}) pair and so on. For example, with *n* = 7, *p* = 5, there will be six values of the (k_{1n}, k_{2n}) pairs. The first set is given by $k_{1n} = k_{2n} = p^{\frac{1}{2}(n-1)} = p^3$, and $x = 2 \times 5^{\frac{1}{2}(7-3)} = 50$. Using Equation (4.2) we get (275,-25) as the other (k_{1n}, k_{2n}) pair.

Similarly, with *s* = 275, *t* = -25, the next (k_{1n}, k_{2n}) pair will be (275,+25) which yields *x* = (275+25)/5 = 60, and Equation (4.2) gives the second pair as (205,95). The third and final duo of the (k_{1n}, k_{2n}) will have (205,-95) so that *x* = 22, and Equation (4.2) gives the final pair as (-29,139). Table 5 shows some examples for *p* = 5 and odd *n* from 3 to 13.

As can be seen from Table 5,

$$x_n = x_{n-2}\,p\,;$$

that is, the recurrence relation generates the final *x* from the *k*-pair of the second last *x*.

The procedure for composites is similar though the analysis is more complicated because of the larger number of (a_1, b_1) couples.

5. Even integers raised to odd powers

For primitive systems, odd-powered even integers would need to have *A*,*B* both odd. However, for odd integers only Class $\bar{1}_4$ contains even powers, so that, since

$$\bar{1}_4 + \bar{1}_4 = \bar{2}_4 \tag{5.1}$$

n	*x*	k_{1n}	k_{2n}
3	2	11 5	-1 5
5	10	55 25	-5 25
	12	55 41	+5 19
7	50	275 125	-25 125
	60	275 205	+25 95
	22	205 -29	-95 139
9	250	1375 625	-125 625
	300	1375 1025	+125 475
	110	1025 -145	-475 695
	168	+145 1199	695 -359
11	1250	6875 3125	-625 3125
	1500	6875 5125	+625 2375
	550	5125 -725	-2375 3475
	840	5995 +725	-1795 3475
	1558	5995 6469	+1795 1321
13	6250	34375 15625	-3125 15625
	7500	34375 25625	+3125 11875
	2750	25625 -3625	-11875 17375
	4200	+3625 29975	17375 -8975
	7790	32345 29975	6605 +8975
	5148	8839 32345	16901 -6605

Table 5: Square components for $p = 5$

and there are no powers at all in Class $\bar{2}_4$, there are no primitive systems where an odd-powered even integer equals a sum of odd squares. For example, $2^3 = 2^2 + 2^2$ which reduces to $2 = 1^2 + 1^2$ (*cf.* [1]). Furthermore, $14^3 = 8 \times 7^3$, but $7 \in \bar{3}_4$, a class for which $N \neq a_1^2 + b_1^2$, so there will not be any (a_{13}, b_{13}) couples for 14^3. All even integers which do not have odd factors reduce to the form

$$2^n = 2^3 2^{n-3} = 2^{n-1}\left(1^2 + 1^2\right). \tag{5.2}$$

When all factors are in Class $\bar{3}_4$ there will be no (a_{13}, b_{13}) couples (as in the odd integer analysis above).

6. Final Comments

The results here indicate how the restrictions to systems which feature sums of squares may be usefully explored. For example, in 4-dimensions the distance of a point from the origin is given by

$$s^2 = x^2 + y^2 + z^2 + t^2. \tag{6.1}$$

We may choose, *n* odd,

$$x^2 + y^2 = X^n$$

and

$$z^2 + t^2 = Y^n,$$

so that

$$s^2 = X^n + Y^n. \tag{6.2}$$

In Z_4, the row, *R*, of an odd square $N \in \bar{1}_4$, is given by [3]:

$$R = 3n(3n \pm 1),\ n = 0,1,2,3,...,\ \ 3 \nmid N \tag{6.3}$$

or

$$R = 2 + 9n(n+1),\ 3 \mid N. \tag{6.4}$$

The reader might like to explore the limitations of Equation (6.2).

References

1. John H Conway & Richard K Guy, *TheBook of Numbers*. New York: Copernicus, 1996.
2. J.V. Leyendekkers, J.M. Rybak & A.G. Shannon, The Characteristics of Primes and Other Integers within the Modular Ring Z_4 and in Class $\bar{1}_4$. *Notes on Number Theory & Discrete Mathematics*. **4 (1)** (1998): 1-17.
3. J.V. Leyendekkers & A.G. Shannon, The Row Structure of Squares in Modular Rings. *Notes on Number Theory & Discrete Mathematics*. 10 **(1)** (2004): 1-11.
4. Neal H. McCoy, *The Theory of Numbers*. New York: Macmillan, 1965, Ch.2.
5. W.G. Nowak. On Sums and Differences of Two Relative Prime Cubes. *Analysis*. **15** (1995): 325-341.

AMS Classification Numbers: 11A41, 11A07