NNTDM 13 (2007), 1, 1-15

A note on rhotrix exponent rule and its applications to some special series and polynomial equations defined over rhotrices

A. Mohammed
Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria
E-mail: abdulmaths@yahoo.com

Abstract

This paper establishes and characterizes the theorem on rhotrix exponent rule, and presents the theory to stimulate systematization of expressing some special series and polynomial equations in terms of the relatively new method of representing arrays of real numbers.

1. Introduction

The theorem on rhotrix exponent rule was first proposed without proof in [1] as part of a note on enrichment exercises through extension to rhotrices. The concept of rhotrices, a new method of representing arrays of real numbers was introduced in [2] as an extension of ideas on matrix-tertions and matrix-noitrets presented in [3] for mathematical enrichment.

It was noted concerning the multiplicative operation (o) defined in [2] that multiplication of rhotrices could be defined in many ways. Following this, an alternative method for multiplication of rhotrices based on its rows and columns vectors was proposed in [4]. The method established some relationships between rhotrices and matrices through an isomorphism.

Therefore, two methods for multiplication of rhotrices are presently available in the literature. Each method provides enabling environment to explore the usefulness of rhotrices as an applicable tool for creation of abstract structures [1] and enrichment of matrix theory [4].

In this paper, we adopt the multiplication in [2] to propose the application of rhotrices as an instrument for expressions of some special series and polynomial equations, after establishing the rhotrix exponent rule[1, *Theorem 5*] and characterizing the exponent laws of multiplication. In addition, we also introduce the definition of a division operator for expressing two rhotrices in quotient form, provided the heart of rhotrix in the denominator is nonzero. This division operator allows series of terms in rhotrices, whose sum approaches a finite value as the number of terms is increased indefinitely, to converge to its sum to infinity.

It is noteworthy to mention that for the purpose of economy in size of this paper, rhotrices of dimension three, considered to be the base rhotrices will be used to present our work. However, all the results stated in this paper hold same for high dimensional rhotrices.

Definition. If R and S are rhotrices of the same size then we define the quotient rhotrix $\frac{R}{S}$ by

$$\frac{R}{S} = RoS^{-1}\text{, provided } h(S) \neq 0. \qquad (1.1)$$

2. The rhotrix exponent rule

We record the following theorem from [1]:
Theorem 2.1.[1, *Theorem 5*]. (*Rhotrix exponent rule).* Let R be a rhotrix as defined in [1]. Then for any integer values of m,

$$R^m = (h(R))^{m-1}\left\langle \begin{array}{ccc} & ma & \\ mb & h(R) & md \\ & me & \end{array} \right\rangle \qquad (2.1)$$

In particular, (i) R^0 is the identity of R and (ii) R^{-1} is the inverse of R.

This theorem which was not proved in [1] will be established in this paper as follows:

Proof: We shall establish this theorem using the principle of mathematical induction. First, we consider the case for positive integer values of m. The result is certainly true for m=1.Now suppose it is true for m=k, so that

$$R^k = (h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle$$

Then we have

$$R^{k+1} = R^K \circ R^1 = (h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

$$\Rightarrow R^{k+1} = (h(R))^{k}\left\langle \begin{array}{ccc} & (k+1)a & \\ (k+1)b & h(R) & (k+1)d \\ & (k+1)e & \end{array} \right\rangle$$

Thus, the theorem holds for the power $(k+1)$, and so it is true for any positive integer. Next, if m is a negative integer, write $m = -k$, so that k is a positive integer. Then by the definition of quotient rhotrix in section 1, we have

$$R^m = R^{-k} = \frac{I}{R^k} = \frac{I}{(h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle} = (h(R))^{-(k-1)}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle^{-1}$$

$$\Rightarrow R^m = R^{-k} = (h(R))^{-(k-1)}.\frac{-1}{(h(R))^2}\left\langle \begin{array}{ccc} & ka & \\ kb & -h(R) & kd \\ & ke & \end{array} \right\rangle = (h(R))^{-k-1}\left\langle \begin{array}{ccc} & -ka & \\ -kb & h(R) & -kd \\ & -ke & \end{array} \right\rangle$$

Provided, $h(R) \neq 0$. Therefore, the theorem holds for all negative values of m.

Finally, if $m = 0$ we have:

$$R^0 = R^{k-k} = R^k \circ R^{-k} = (h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle \circ (h(R))^{-k-1}\left\langle \begin{array}{ccc} & -ka & \\ -kb & h(R) & -kd \\ & -ke & \end{array} \right\rangle$$

$$\Rightarrow R^0 = \frac{1}{(h(R))}\left\langle \begin{array}{ccc} & (k-k)a & \\ (k-k)b & h(R) & (k-k)d \\ & (k-k)e & \end{array} \right\rangle$$

$$\Rightarrow R^0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Thus, the theorem holds for the power $m = 0$.Hence, the theorem is true for all integer values of m.

As a corollary to the theorem on rhotrix exponent rule, we have the following:

Corollary 2.2. If $m = 0$ and $m = -1$ in theorem 2.1 then
(i) R^0 is the identity of rhotrix R and
(ii) R^{-1} is the inverse of rhotrix R respectively.

Proof. (i) If $m = 0$ in theorem 2.1, we have:

$$R^0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Therefore, $R \circ R^0 = R^0 \circ R = R$

Hence, R^0 is the identity of rhotrix R.

(ii) If $m = -1$ in theorem 2.1, we have:

$$R^{-1} = (h(R))^{-2}\left\langle \begin{array}{ccc} & -a & \\ -b & h(R) & -d \\ & -e & \end{array} \right\rangle = \frac{-1}{(h(R))^2}\left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle$$

Therefore, $R \circ R^{-1} = R^{-1} \circ R = I$
Thus, R^{-1} is the inverse of rhotrix R.

The following corollary is obvious:

Corollary 2.3. If R is a unit heart rhotrix then we have for any integer values of m,

$$R^m = \left\langle \begin{array}{ccc} & ma & \\ mb & 1 & md \\ & me & \end{array} \right\rangle$$

Proof. Substituting $h(R) = 1$ in equation (2.1), then the result follows.

Properties of the rhotrix exponent rule: If m and n are integers, then for any real rhotrix R

(a) $R^m \circ R^n = R^{m+n}$

(b) $\dfrac{R^m}{R^n} = R^{m-n}, provided, h(R) \neq 0$

(c) $\left(R^m\right)^{1/n} = R^{\frac{m}{n}}$

(d) $\left(R^m\right)^n = R^{mn}$

(e) $(kR)^m = k^m R^m$ (Where k is a scalar)

(f) $R^0 = I$ (Where I is the identity of R)

(g) $R^{-1} = \dfrac{I}{R} = I \circ R^{-1}, provided, h(R) \neq 0$

(h) $R^m = 0$ (or zero rhotrix), provided $h(R) = 0$ and m≥2

3. Some special series in terms of rhotrices

The rhotrix exponent rule and its above properties enables expression of algebraic series and expansion in terms of rhotrices.

Arithmetic series in terms of rhotrices: Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ and $D = \left\langle \begin{matrix} & f & \\ g & h(D) & i \\ & j & \end{matrix} \right\rangle$

be any real rhotrices of the same size. Consider the series S_n defined in terms of rhotrices R and D as follows:

$$S_n = \sum_{k=1}^{n} \left[R + (n-1)D \right]$$

Obviously, S_n is an arithmetic progression having rhotrices R and D, respectively, as its first term and common difference. Now, we can apply the method in [5] to obtain the n[th] sum as follows:

$$S_n = \frac{n}{2}\{2R + (n-1)D\} = \frac{n}{2}\left\{ 2\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + (n-1)\left\langle \begin{matrix} & f & \\ g & h(D) & i \\ & j & \end{matrix} \right\rangle \right\}$$

$$\Rightarrow S_n = \left\langle \begin{matrix} & \frac{2na + n(n-1)f}{2} & \\ \frac{2nb + n(n-1)g}{2} & \frac{2nh(R) + n(n-1)h(D)}{2} & \frac{2nd + n(n-1)i}{2} \\ & \frac{2ne + n(n-1)j}{2} & \end{matrix} \right\rangle \quad (3.1)$$

For example, using equation (3.1), the sum of the first twenty terms of the following arithmetic series:

$$S_n = \left\langle \begin{matrix} & 3 & \\ -2 & 5 & 1 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 5 & \\ -1 & 2 & 5 \\ & 2 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 7 & \\ 0 & -1 & 9 \\ & 4 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 9 & \\ 1 & -4 & 13 \\ & 6 & \end{matrix} \right\rangle + ----$$

with first term, $R = \left\langle \begin{matrix} & 3 & \\ -2 & 5 & 1 \\ & 0 & \end{matrix} \right\rangle$ and common difference, $D = \left\langle \begin{matrix} & 2 & \\ 1 & -3 & 4 \\ & 2 & \end{matrix} \right\rangle$ is given by:

$$S_{20} = \left\langle \begin{array}{ccc} & 440 & \\ 150 & -470 & 780 \\ & 380 & \end{array} \right\rangle$$

Remark 1. This is the same as separating each term of the series and summing individually.

Analogously, we can now have the following:

Geometric series in terms of rhotrices: Let us consider a geometric series in terms of real rhotrices $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $T = \left\langle \begin{array}{ccc} & f & \\ g & h(T) & i \\ & j & \end{array} \right\rangle$ as follows:

$$U_n = \sum_{k=1}^{n} R \circ T^{k-1} \qquad (3.2)$$

Obviously, R and T denote, respectively, the first term and common ratio of U_n. Now, using the method in [5], we obtain by computation the nth sum:

$$U_n = R \circ \left(\frac{I - T^n}{I - T} \right) \qquad (3.3)$$

$$\Rightarrow U_n = R \circ \left\{ \left(I - T^n \right) \circ \left(I - T \right)^{-1} \right\}, \mathit{provided}, h(T) \neq 1 \qquad (3.4)$$

Remark 2. If $h(T) = 1$, then the quotient rhotrix within the bracket on the right hand side of equation (3.3) becomes undefined, because of the singularity of expression $(I - T)$, see equation (1.1) in the definition of quotient rhotrix. It is also necessary to note that the dimension of identity rhotrix *I* correspond to that of rhotrices R and T in equation (3.4).

For instance, the sum of the first tenth terms of the geometric series with first term $R = \left\langle \begin{array}{ccc} & 5 & \\ -3 & 4 & 1 \\ & 2 & \end{array} \right\rangle$ and common ratio $T = \left\langle \begin{array}{ccc} & 0 & \\ 4 & 2 & 1 \\ & -3 & \end{array} \right\rangle$ is given by

$$U_{10} = \sum_{k=1}^{10} R \circ T^{k-1} = R \circ \left\{ \left(I - T^{10} \right) \circ \left(I - T \right)^{-1} \right\} = \left\langle \begin{array}{ccc} & 5115 & \\ 62483 & 4092 & 17411 \\ & -47118 & \end{array} \right\rangle$$

Theorem3.1. Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be a non-unit heart rhotrix then the finite series

$$\sum_{k=1}^{n} R^{k-1} = \left\{ n(h(R))^{n-1}(1-h(R))^{-1} - \left(\frac{1-(h(R))^n}{(1-h(R))^2} \right) \right\} \left\langle \begin{matrix} & -a & \\ -b & \dfrac{1-(h(R))^n}{\left\{ n(h(R))^{n-1} - \left(\dfrac{1-(h(R))^n}{1-h(R)} \right) \right\}} & -d \\ & -e & \end{matrix} \right\rangle$$

provided, $h(R) \neq 1$.

Proof: Given the rhotrix $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ such that $h(R) \neq 1$.

Let

$$G_n = \sum_{k=1}^{n} R^{k-1} \qquad (3.5)$$

Multiplying equation (3.5) by rhotrix R and applying the rhotrix exponent properties in section 2 above to get:

$$R \circ G_n = \sum_{k=1}^{n} R^{k} \qquad (3.6)$$

Subtracting (3.6) from (3.5), all the terms cancels out except for two, we obtain

$$G_n - R \circ G_n = I - R^n$$

$$\Rightarrow G_n \circ (I - R) = I - R^n$$

$$\Rightarrow G_n = \frac{I - R^n}{I - R} \qquad \text{Provided, } \quad R \neq I \qquad (3.7)$$

$$\Rightarrow G_n = \left(I - R^n\right) \circ \left(I - R\right)^{-1} \qquad \text{Provided, } h(R) \neq 1 \qquad (3.8)$$

Substituting the expressions for rhotrices I and R in equation (3.8):

$$\Rightarrow G_n = \left\langle \begin{matrix} & -na(h(R))^{n-1} & \\ -nb(h(R))^{n-1} & 1-(h(R))^n & -nd(h(R))^{n-1} \\ & -ne(h(R))^{n-1} & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a(1-h(R))^{-2} & \\ b(1-h(R))^{-2} & (1-h(R))^{-1} & d(1-h(R))^{-2} \\ & e(1-h(R))^{-2} & \end{matrix} \right\rangle$$

Hence,

$$G_n = \left\{ n(h(R))^{n-1}(1-h(R))^{-1} - \left(\frac{1-(h(R))^n}{(1-h(R))^2} \right) \right\} \left\langle \begin{matrix} & -a & \\ -b & \dfrac{1-(h(R))^n}{\left\{ n(h(R))^{n-1} - \left(\dfrac{1-(h(R))^n}{1-h(R)} \right) \right\}} & -d \\ & -e & \end{matrix} \right\rangle$$

Provided, $h(R) \neq 1$.

Remark 3. If $h(R) = 1$ in *theorem 2,* then the nth partial sum of the finite series is an undefined quotient rhotrix, obtainable by substituting $h(R) = 1$ in the expression of R in equation (3.7).

Corollary 3.2. If $h(R) = 0$ then $\sum_{k=1}^{n} \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle^{k-1} = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$

Proof. Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$, and let

$$G_n = \sum_{k=1}^{n} R^{k-1} = R^0 + R^1 + R^2 + R^3 + --- + R^{n-1}$$

Since $h(R) = 0$, it follows from the rhotrix exponent property (h) in section 2 that any term in G_n having index ≥2 becomes zero rhotrix. Thus,

$$G_n = R^0 + R^1 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$$

Theorem 3.3. Let $R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be a rhotrix such that $0<|h(R)|<1$ then the infinite series

$$\sum_{k=1}^{\infty} R^{k-1} = \frac{1}{(1-h(R))^2}\left\langle \begin{matrix} & a & \\ b & 1-h(R) & d \\ & e & \end{matrix} \right\rangle$$

Proof. Let us rewrite the infinite series $\sum_{k=1}^{\infty} R^{k-1}$ as $\lim_{n\to\infty}\left(\sum_{k=1}^{n} R^{k-1}\right)$. Now, using equation (3.8), we have:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} R^{k-1}\right)=\lim_{n\to\infty}\left\{(I-R^n)\circ(I-R)^{-1}\right\} \qquad (3.9)$$

Since it was given that $0<|h(R)|<1$, then we can apply the corollary in [2] on the limit in equation (3.9) to get:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} R^{k-1}\right)=\lim_{n\to\infty}\left\{(I-R^n)\circ(I-R)^{-1}\right\}=\left\{(I-0)\circ(I-R)^{-1}\right\}=\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle\circ\left\langle \begin{matrix} & -a & \\ -b & 1-h(R) & e \\ & -d & \end{matrix} \right\rangle^{-1}$$

on substituting the expressions for I and R.

Hence,

$$\sum_{k=1}^{\infty} R^{k-1} = \frac{1}{(1-h(R))^2}\left\langle \begin{matrix} & a & \\ b & 1-h(R) & d \\ & e & \end{matrix} \right\rangle, \text{ provided } \quad 0<|h(R)|<1.$$

Proposition 3.4. If $h(T)=-1$ and $n\in 2Z$ in the finite series $S_n=\sum_{k=1}^{n} R\circ T^{k-1}$ then S_n is singular.

Proof. To establish this result, it is sufficient for us to show that the heart of the nth partial sum of the series is equal to zero (i.e. $h(S_n)=0$) with the constraints conditions.

Suppose R and T , respectively, are $R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ and $T=\left\langle \begin{matrix} & f & \\ g & h(T) & i \\ & j & \end{matrix} \right\rangle$

in the geometric series $S_n=\sum_{k=1}^{n} R\circ T^{k-1}$. Using equation (3.4), we have

$$S_n = R \circ \left\{ \left(I - T^n\right) \circ \left(I - T\right)^{-1} \right\}$$

Substituting the expressions for R,T and I,

$$\Rightarrow S_n = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\{ \begin{array}{l} \left\langle \begin{array}{ccc} & -nf(h(T))^{n-1} & \\ -ng(h(T))^{n-1} & 1-(h(T))^{n} & -ni(h(T))^{n-1} \\ & -nj(h(T))^{n-1} & \end{array} \right\rangle \\ \circ \left\langle \begin{array}{ccc} & f(1-h(T))^{-2} & \\ g(1-h(T))^{-2} & (1-h(T))^{-1} & i(1-h(T))^{-2} \\ & j(1-h(T))^{-2} & \end{array} \right\rangle \end{array} \right\} \tag{3.10}$$

Therefore, substituting $h(T) = -1$ in equation (3.10) and taking cognizance of $n \in 2\mathrm{Z}$

$$\Rightarrow S_n = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\{ \left\langle \begin{array}{ccc} & nf & \\ ng & 0 & ni \\ & nj & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2^{-2} f & \\ 2^{-2} g & 2^{-1} & 2^{-2} i \\ & 2^{-2} j & \end{array} \right\rangle \right\}$$

By associativity of multiplicative operation $(\circ)$, it follows that

$$S_n = \frac{h(R)}{2} \left\langle \begin{array}{ccc} & nf & \\ ng & 0 & ni \\ & nj & \end{array} \right\rangle$$

$$\Rightarrow h\left(S_n\right) = 0$$

Hence, S_n is a singular rhotrix provided $h(T) = -1,$ and $n \in 2\mathrm{Z}$.

Theorem 3.5. Let $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ be non-zero and non-unit heart rhotrix then the finite series

$$\sum_{k=1}^{n} \left(\frac{I}{R} \right)^{k-1} = \left(R^{n+1} - R\right) \circ (R^{n+1} - R^{n})^{-1} \qquad \text{Provided } h(R) \in \Re - \{0,1\}.$$

Proof. Let $G_n = \sum_{k=1}^{n} \left(\frac{I}{R}\right)^{k-1}$ (3.11)

Multiplying equation (3.11) by rhotrix $\frac{I}{R}$ and applying the rhotrix exponent properties in section 2 above to get:

$$\frac{I}{R} \circ G_n = \sum_{k=1}^{n} \left(\frac{I}{R}\right)^{k-1} \qquad (3.12)$$

Subtracting (3.12) from (3.11), all the terms cancels out except for two, we obtain

$$G_n = \frac{I - \left(\frac{I}{R}\right)^n}{I - \left(\frac{I}{R}\right)} = \left(\frac{R^n - I}{R^n}\right) \circ \left(\frac{R - I}{R}\right)^{-1} = \left(\frac{R^n - I}{R^n}\right) \circ \left(\frac{R}{R - I}\right) = \frac{R^{n+1} - R}{R^{n+1} - R^n}$$

$$\Rightarrow G_n = \left(R^{n+1} - R\right) \circ \left(R^{n+1} - R^n\right)^{-1}, provided, h(R) \in \Re - \{0,1\}$$

Remark 4. If $h(R) = 0$, the finite series in theorem 3.5 becomes undefined.

Theorem 3.6. Let $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ be non-zero and non-unit heart rhotrix then the infinite series

$$\sum_{k=1}^{\infty} \left(\frac{I}{R}\right)^{k-1} = \frac{R}{R - I} \quad \text{provided } |h(R)| > 1.$$

Proof. Let us rewrite the given infinite series as follows:

$$\sum_{k=1}^{\infty} \left(\frac{I}{R}\right)^{k-1} = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \left(\frac{I}{R}\right)^{k-1} \right\}$$

Using the result in theorem 3.5 and the rhotrix exponent properties in section 2, we have:

$$\sum_{k=1}^{\infty} \left(\frac{I}{R}\right)^{k-1} = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \left(\frac{I}{R}\right)^{k-1} \right\} = \lim_{n \to \infty} \left\{ \left(R^{n+1} - R\right) \circ (R^{n+1} - R^n)^{-1} \right\}$$

$$= \lim_{n\to\infty}\left\{\frac{R^{n+1}-R}{R^{n+1}-R^{n}}\right\} = \lim_{n\to\infty}\left\{\frac{R^{n}(R-R^{1-n})}{R^{n}(R-I)}\right\} = \lim_{n\to\infty}\left\{\frac{R-R^{1-n}}{R-I}\right\}$$

$$= \left(\frac{R}{R-I}\right) - \left(\frac{R}{R-I}\right)\circ\left(\lim_{n\to\infty} R^{-n}\right) = \left(\frac{R}{R-I}\right) - \left(\frac{R}{R-I}\right)\circ(0)$$

$$= \frac{R}{R-I}, \, provided, |h(R)| > 1.$$

Binomial series for rhotrix of positive exponent: The theorem on Binomial expansion in [5] can be extended to exponent rhotrices as follows: If P and Q are any two rhotrices of the same size then for any $n \in \mathbb{Z}^{+}$

$$[P+Q]^{n} = P^{n} + {}^{n}C_{1}\left(P^{n-1}\circ Q\right) + {}^{n}C_{2}\left(P^{n-2}\circ Q^{2}\right) + {}^{n}C_{3}\left(P^{n-3}\circ Q^{3}\right) + \ldots + Q^{n}, where, {}^{n}C_{r} = \frac{n!}{(n-1)!\,r!}$$

The Binomial theorem enable us to perceive that any rhotrix of positive exponent n can be expressed in terms of two invertible rhotrices P and Q such that $R^{n} = (P+Q)^{n}$.

For instance, if

$$R^{n} = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle^{n}$$

then we can rewrite

$$R^{n} = \left[\left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle + \left((-1)\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \right)\right]^{n}$$

So that by Binomial theorem

$$\left[\left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle + \left((-1)\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \right)\right]^{n} = \left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle^{n} - n\left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle^{n-1}$$

$$+ \frac{n(n-1)}{2!}\left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle^{n-2} - \frac{n(n-1)(n-2)}{3!}\left\langle \begin{matrix} & a & \\ b & h(R)+1 & d \\ & e & \end{matrix} \right\rangle^{n-3} + \text{-----} + (-1)^{n}\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

(3.13)

Now, if we denote $\Delta = \left\langle \begin{array}{ccc} & a & \\ b & h(R)+1 & d \\ & e & \end{array} \right\rangle$ in equation (3.13), then it follows that

$$R^n = \Delta^n - n\Delta^{n-1} + \frac{n(n-1)}{2!}\Delta^{n-2} - \frac{n(n-1)(n-2)}{3!}\Delta^{n-3} + - - - - - - - - + (-1)^n I$$

is a Binomial series for R^n. Where I is the identity of rhotrix R.

4. The rhotrix polynomial equations

Polynomial equations can be defined over variable(s) and coefficients which are themselves rhotrices. For instance if A and B are real rhotrices of the same size then the equations and $X + A = B$ and $C \circ Y = D$ have unique solutions $X = B - A$ and $Y = D \circ C^{-1}$ respectively, provided $h(C) \neq 0$.

Theorem 4.1. Let A, B and C be non-singular rhotrices of the same size such that $[h(B)]^2 - 4h(A)h(C) \geq 0$ then there exist non-singular rhotrix X satisfying the polynomial equation

$$A \circ X^2 + B \circ X + C = 0.$$

Proof. Let $A = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & h(A) & a_4 \\ & a_5 & \end{array} \right\rangle$, $B = \left\langle \begin{array}{ccc} & b_1 & \\ b_2 & h(B) & b_4 \\ & b_5 & \end{array} \right\rangle$, and $C = \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & h(C) & c_4 \\ & c_5 & \end{array} \right\rangle$ in the given polynomial equation

$$A \circ X^2 + B \circ X + C = 0 \qquad (4.1)$$

Then we need to find two values for the rhotrix

$$X = \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & h(X) & x_4 \\ & x_5 & \end{array} \right\rangle$$

satisfying the quadratic polynomial equation (4.1).

Now, if we rewrite equation (4.1) as

$$\left\langle \begin{array}{ccc} & a_1 & \\ a_2 & h(A) & a_4 \\ & a_5 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & h(X) & x_4 \\ & x_5 & \end{array} \right\rangle^2 + \left\langle \begin{array}{ccc} & b_1 & \\ b_2 & h(B) & b_4 \\ & b_5 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & h(X) & x_4 \\ & x_5 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & h(C) & c_4 \\ & c_5 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\Rightarrow\left\langle \begin{array}{ccc} & \begin{array}{l} a_1(h(X))^2 \\ +2h(A)h(X)x_1 \\ +b_1h(X)+x_1h(B)+c_1 \end{array} & \\ \begin{array}{l} a_2(h(X))^2 \\ +2h(A)h(X)x_2 \\ +b_2h(X)+x_2h(B)+c_2 \end{array} & \begin{array}{l} h(A)(h(X))^2 \\ +h(B)h(X)+h(C) \end{array} & \begin{array}{l} a_4(h(X))^2 \\ +2h(A)h(X)x_4 \\ +b_4h(X)+x_4h(B)+c_4 \end{array} \\ & \begin{array}{l} a_5(h(X))^2 \\ +2h(A)h(X)x_5 \\ +b_5h(X)+x_5h(B)+c_5 \end{array} & \end{array}\right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array}\right\rangle \quad (4.2)$$

From equation (4.2), we have the following simultaneous equations:

$$a_1(h(X))^2 + 2h(A)h(X)x_1 + b_1h(X) + x_1h(B) + c_1 = 0 \quad (4.3)$$
$$a_2(h(X))^2 + 2h(A)h(X)x_2 + b_2h(X) + x_2h(B) + c_2 = 0 \quad (4.4)$$
$$h(A)(h(X))^2 + h(B)h(X) + h(C) = 0 \quad (4.5)$$
$$a_4(h(X))^2 + 2h(A)h(X)x_4 + b_4h(X) + x_4h(B) + c_4 = 0 \quad (4.6)$$
$$a_5(h(X))^2 + 2h(A)h(X)x_5 + b_5h(X) + x_5h(B) + c_5 = 0 \quad (4.7)$$

From equation (4.5), we have a quadratic equation in terms of h(X). Therefore, the two roots of h(X) are:

$$h(X) = \frac{-h(B) + \sqrt{(h(B))^2 - 4h(A)h(C)}}{2h(A)} \quad (4.8)$$

or

$$h(X) = \frac{-h(B) - \sqrt{(h(B))^2 - 4h(A)h(C)}}{2h(A)} \quad (4.9)$$

Note that $h(A) \neq 0$ and $(h(B))^2 - 4h(A)h(B) \geq 0$.
Now, from equation (4.3), we have

$$x_1 = -\frac{a_1(h(X))^2 + b_1h(X) + c_1}{2h(A)h(X) + h(B)} \quad (4.10)$$

Similarly, from equations (4.4), (4.6) and (4.7) we have respectively

$$x_2 = -\frac{a_2(h(X))^2 + b_2h(X) + c_2}{2h(A)h(X) + h(B)} \quad (4.11)$$

$$x_4 = -\frac{a_4(h(X))^2 + b_4h(X) + c_4}{2h(A)h(X) + h(B)} \quad (4.12)$$

and

$$x_5 = -\frac{a_5(h(X))^2 + b_5 h(X) + c_5}{2h(A)h(X) + h(B)} \qquad (4.13)$$

Therefore, by substituting equation (4.8) in equations (4.10), (4.11), (4.12) and (4.13) then we obtain the first value of rhotrix X satisfying the polynomial equation (4.1). Similarly, by substituting equation (4.9) in equations (4.10), (4.11), (4.12) and (4.13) then we also obtain the second value of rhotrix X satisfying the polynomial equation (4.1). Hence, the result follows.

Remark 5. The polynomial equation of degree two in theorem 4.1 can be generalized to degree n.

Acknowledgement

My sincere thanks are due to my research supervisors, Professor G.U. Garba and Dr. B. Sani, for their helpful suggestions and encouragement. I also wish to thank my employer, Ahmadu Bello University, Zaria, Nigeria for funding this relatively new area of research.

References

[1] MOHAMMED, A., (2007), *Enrichment exercises through extension to rhotrices,* Int. J. Math. Educ. Sci. Tech., **38,** 131-136.

[2] AJIBADE, A.O., (2003), *The concept of rhotrix for mathematical enrichment,* Int. J. Math. Educ. Sci. Tech., **34,** 175-179.

[3] ATANASSOV, K.T. and SHANNON, A,G., (1998), *Matrix-tertions and matrix-noitrets: exercises in mathematical enrichment,* Int. J. Math. Educ. Sci. Tech., **29,** 898-903.

[4] SANI, B., (2004), *An alternative method for multiplication of rhotrices,* Int. J. Math. Educ. Sci. Tech., **35,** 777-781.

[5] BACKHOUSE, J.K. and HOULDSWORTH, S.P.T., (1982), *Pure mathematics: a first course* (Harlow,Essex:Longman Group Limited).