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On Steiner Loops of cardinality 20

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Abstract: *It is well known that there are five classes of sloops of cardinality* 16 " **SL**(16)*s*" *according to the number of sub*-**SL**(8)*s* [4, 6]. *In this article*, *we will show that there are exactly* 8 *classes of nonsimple sloops and* 6 *classes of simple sloops of cardinality* 20 "**SL**(20)*s*". *Based on the cardinality and the number of* (*normal*) *subsloops of* **SL**(20*), we will construct in section 3 all possible classes of nonsimple* **SL**(20)*s and in section* 4 *all possible classes of simple* **SL**(20)s. *We exhibit the algebraic and combinatoric properties of* **SL**(20)*s to distinguish each class.*

So we may say that there are six classes of **SL**(20)*s having one sub*-**SL**(10) *and n sub*-**SL**(8)s *for n* = 0, 1, 2, 3, 4 *or* 6. *All these sloops are subdirectly irreducible having exactly one proper homomorphic image isomorphic to* **SL**(2). *For n* = 0, *the associated* **SL**(20) *is a nonsimple subdirectly irreducible having one sub*-**SL**(10) *and no sub*-**SL**(8)*s. Indeed, the associated Steiner quasigroup* **SQ**(19) *of this case supplies us with a new example for a semi-planar* **SQ**(19), *where the smallest well-known example of semi-planar squags is of cardinality* 21 *" cf.* [3]*".*

It is well known that there is a class of planar Steiner triple systems (**STS**(19)s) *due to Doyen* [7], *where the associated planar* **SL**(20) *has no sub*-**SL**(10) *and no sub*-**SL**(8). *In section 4 we will show that there are other* 6 *classes of simple* **SL**(20)*s having n sub*-**SL**(8)*s for n* = 0, 1, 2, 3, 4, 6, *but no sub*-**SL**(10)*s*. *It is well-known that a sub*-**SL**(*m*) *of an* **SL**(2*m*) *is normal. In the last theorem of this section, we give a necessary and sufficient condition for a sub*-**SL**(2) *to be normal of an* **SL**(2*m*). *Accordingly*, *we have shown that if a sloop* **SL**(20) *has a sub*-**SL**(10) and 12 *sub*-**SL**(8), *then this sloop is isomorphic to the direct product* **SL**(10) × **SL**(2) *and if a sloop* **SL**(20) *has* 12 *sub*-**SL**(8)*s and no sub*-**SL**(10)*, then this sloop is a subdirectly irreducible having exactly one proper homomorphic image isomorphic to* **SL**(10). *In section* 5, *we describe how can one construct an example for each class of smiple and of nonsimple* **SL**(20)s.

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Introduction:

A Steiner loop (briefly sloop) is a groupoid $\boldsymbol{S} = (S; \cdot, 1)$ with neutral element 1 satisfying the identities:

$$x \cdot x = 1 \quad , \quad x \cdot y = y \cdot x \quad , \quad x \cdot (x \cdot y) = y .$$

A Steiner quasigroup (briefly squag) is a groupoid $\boldsymbol{Q} = (Q; *)$ satisfying the identities:

$$x * x = x \quad , \quad x * y = y * x \quad , \quad x * (x * y) = y \, .$$

Notice that both squags and sloops are quasigroups [5, 8].

We use the abbreviations **SL**(n) and **SQ**(n) for a sloop and a squag of cardinality n, respectively. A sloop is called Boolean (or Boolean group) if it satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

A Steiner triple system is a pair (P; B), where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points p_1, $p_2 \in P$, there is a unique block $b \in B$ with $\{p_1, p_2\} \subseteq b$. If the cardinality of the set of points P is equal to n, the Steiner triple system (P; B) will be denoted by **STS**(n). It is well known that a necessary and sufficient condition for the existence of an **STS**(n) is $n \equiv 1$ or 3 (mod 6) [8, 11]. There is a one to one correspondence between sloops (squags) and Steiner triple systems given by the relation:

$x \cdot y = z \Leftrightarrow \{x, y, z\}$ is a block [8, 11, 12].

Quackenbush [12] proved that the congruences of sloops are permutable, regular, and Lagrangian. A subsloop ***S*** of a sloop ***L*** is called normal iff $(x \cdot y) \cdot S = x \cdot (y \cdot S)$ for all $x, y \in L$. Also in [12] was proved that if ***S*** is a subsloop of ***L*** and |L| = 2 |S|, then ***S*** is normal.

A three distinct points x, y, z are called a triangle if $\{x, y, z\}$ does not form a block (or equivalently, if $\{x, y, z\}$ does not contains the identity element and $x \cdot y \neq z$). An **STS** is planar if it is generated by every triangle (. A planar **STS**(n) exists for each $n \geq 7$ and $n \equiv 1$ or 3 (mod 6) [7]). The associated squag and sloop of a planar triple system are also called planar. Quackenbush [12] has shown that the only nonsimple finite planar sloop (squag) has 8 (9) elements. Accordingly, we may say that there is always a simple **SQ**(n) and a simple **SL**(n +1) for all $n > 9$ and $n \equiv 1$ or 3 (mod 6).

A semi-planar sloop (squag) is a simple sloop (squag) each of whose triangles generates either the whole sloop (squag) or a sub-**SL**(8) (a sub-**SQ**(9)). The associated **STS** with a semi-planar sloop (squag) is said to be a semi-planar **STS** or more precisely a semi-7-planar **STS** (a semi-9-planar **STS**), if each of whose triangles generates either the whole **STS** or a sub-**STS**(7) (a sub-**STS**(9)) [2, 3, 12].

The author in [2, 3] has given a construction of semi-planar sloops of cardinality 2(n + 1) and a construction of semi-planar squags of cardinality 3 n for each possible $n > 3$.
An extensive study of sloops can be found in [5, 8, 11]. We will use in this article some basic concepts of universal algebra [9] and some other concepts of graph theory [10].

There is a well-known classification of all **SL**(16)s based on the number of sub-**SL**(8)s. In fact, there are four classes from five classes of **SL**(16)s are subdirectly irreducible [4, 6, 11]. The next admissible order for sloops is of cardinality 20. In this article we are not interesting in counting the number of distinct **SL**(20)s, but we exhibit the algebraic and combinatoric properties of **SL**(20)s to distinguish each class.

We may divide all **SL**(20)s based on the cardinality and the number of the (normal) subsloops of an **SL**(20) into the following classes. Note that a sub-**SL**(10) in an **SL**(20) is always normal.

1- There are planar **STS**(19)s due to Doyen [7]. The associated planar sloops **SL**(20)s and the associated planar squags **SQ**(19)s are simple. Indeed, these planar **SL**(20)s (or planar **SQ**(19)s) have no nontrivial subsloops (subsquags).
2- In section 3, we will show that there are 7 classes of **SL**(20)s. Each of these classes has exactly one sub-**SL**(10) and n sub-**SL**(8)s (for n = 0, 1, 2, 3, 4, 6, 12). For n = 12, we have a class of **SL**(20)s containing one sub-**SL**(10) and 12 sub-**SL**(8)s, we will show that this sloop must be isomorphic to the direct product **SL**(10) × **SL**(2). Also, all **SL**(20)s of the other six classes (for n = 0, 1, 2, 3, 4, 6) are nonsimple subdirectly irreducible.
   For n = 0, the associated **SL**(20) is nonsimple subdirectly irreducible having one sub-**SL**(10) and no sub-**SL**(8)s. Indeed, the associated **SQ**(19) with this case supplies us with a new example for a semi-planar **SQ**(19) (or a semi-9-planar **STS**(19)). Of course, this class of semi-planar **SQ**(19)s are not planar and have exactly one sub-**SQ**(9), but no sub-**SQ**(7). It will be convenient to note at this point that the smallest well-known example of semi-planar squags is of cardinality 21 " cf. [3]". An example of this new case (a semi-planar **SQ**(19)) will be given in section 5.
3- In section 4, we will show that there are five classes of semi-planar **SL**(20)s based on the number n = 1, 2, 3, 4 or 6 of sub-**SL**(8)s. All of these semi-planar **SL**(20)s are simple and not planar. In addition, the associated **SQ**(19)s of these classes are simple and each triangle generates a sub-**SQ**(7) or the whole **SQ**(19).
4- According to the construction given in section 4, there is a class of **SL**(20)s having no sub-**SL**(10) and 12 sub-**SL**(8)s. We will show that these sloops are subdirectly irreducible and have exactly one proper congruence with classes of cardinality two (one proper homomorphic image isomorphic to **SL**(10)).

In fact, these are all classes of **SL**(20)s. In section 5 we describe how can one construct an example for each class.

2. Construction of an SL($2n$) = 2 $\otimes_\alpha L_1$.

Using the doubling construction **SL**($2n$)s [11], we will study in this section some properties of subsloops of **SL**($2n$)s .

Let $\boldsymbol{T}_1 = (P^*_1; B_1)$ be an **STS**(n) and its corresponding sloop $\boldsymbol{L_1} = (P_1; \cdot, e)$, where $P^*_1 = \{a_1, \dots, a_n\}$ and $P_1 = P^*_1 \cup \{e\}$. Consider the set of 1-factors defined by $F_i = \{a_l\, a_k : a_l \,.\, a_k = a_i$ and $a_i, a_l, a_k \in P_1\}$, then the class $\boldsymbol{F} = \{F_1, F_2, \dots, F_n\}$ forms a 1-factorization of the complete graph $\mathbf{K}_n$ on the set of vertices P_1.

By taking the set $P_2 = \{b, b_1, b_2, \dots, b_n\}$ with $P_1 \cap P_2 = \varnothing$ and $G_i = \{b\, b_i\} \cup \{b_l\, b_k : a_l \cdot a_k = a_i$ for $i \notin \{l, k\}\}$, then the class of 1-factors $\boldsymbol{G} = \{G_1, G_2, \dots, G_n\}$ forms a 1-factorization of the complete graph $\mathbf{K}_n$ on the set of vertices P_2. There is a well-known construction of an **STS**($2n+1$) = $(P^*; B)$ [11], where $P^* = P^*_1 \cup P_2$ and the set of triples $B = B_1 \cup \{\{b_l, b_k, a_i\} : b_l\, b_k \in G_{\alpha(i)}\}$ for any permutation α on the set $\{1, \dots, n\}$. The constructed **STS**($2n+1$) = $(P^*; B)$ and the associated sloop **SL**($2n+2$) = $(P; \cdot, e)$ will be denoted by $2 \otimes_\alpha \boldsymbol{T_1}$ and $2 \otimes_\alpha \boldsymbol{L_1}$, respectively.

If we choose the permutation α = the identity, then the constructed sloop $\boldsymbol{L} = 2 \otimes_\alpha \boldsymbol{L_1}$ isomorphic to the direct product of **SL**(n+1) = $\boldsymbol{L_1}$ and the 2-element sloop **SL**(2). We observe that $\boldsymbol{L_1}$ is a normal subsloop of $2 \otimes_\alpha \boldsymbol{L_1}$ for any permutation α.

It is easy to prove the following fact.

Lemma 1. *Let* $2 \otimes_\alpha \boldsymbol{L_1} = (P = P_1 \cup P_2; \cdot, e)$ *be the constructed sloop of cardinality* $2n$ *with the subsloop* $\boldsymbol{L_1} = (P_1; \cdot, e)$ *of cardinality n. Then any subsloop* **S** *of* **L** *with* $S - P_1 \neq \varnothing$ *satisfies* $|S \cap P_1| = (1/2)|S|$.

We note that if $\boldsymbol{L_1}$ is a planar sloop, then $|S \cap P_1| = (1/2)|S| = 1, 2$ or 4.

In the following we consider the **SL(**10) = $\boldsymbol{L_1} = (P_1 = P^*_1 \cup \{e\}; \cdot, e)$ associated with the **STS**(9) = $(P^*_1; B_1)$, where $P^*_1 = \{a_1, \dots, a_9\}$. Also, we consider the set $P_2 = \{b, b_1, b_2, \dots, b_9\}$ with $P_1 \cap P_2 = \varnothing$ and α be any permutation on the set $N = \{1, 2, \dots, 9\}$ to construct the **SL**(20) = $2 \otimes_\alpha \boldsymbol{L_1}$. Moreover, we consider the **STS**(9) = $(N; X)$, where X is defined by: $\{i, j, k\} \in X$ if and only if $\{a_i, a_j, a_k\} \in B_1$.

The constructed **STS**(19) is given by $2 \otimes_\alpha \boldsymbol{T_1} = (\mathrm{P}^* = P^*_1 \cup P_2; B = B_1 \cup B_{12})$, where $B_{12} = \{\{a_i, b_j, b_k\} : b_j\, b_k \in G_{\alpha(i)}\}$. Let $2 \otimes_\alpha \boldsymbol{L_1} = (P; \cdot, e)$ be the associated sloop with **STS**(19) = $2 \otimes_\alpha \boldsymbol{T_1}$.

For each block $\{a_i, a_j, a_k\} \in B_1$ there is a sub-1-factorization $f = \{f_i = \{e\, a_i, a_j\, a_k\}, f_j = \{e\, a_j, a_i\, a_k\}, f_k = \{e\, a_k, a_i\, a_j\}\}$ of F. Conversely, if there is a sub-1-factorization on the 4-element subset $\{e, a_i, a_j, a_k\}$, then $\{a_i, a_j, a_k\}$ is a block in B_1. This means that there is a one-one correspondence between the set of blocks of B_1 and the sub-1-factorizations of $\mathbf{K}_4$ in $\boldsymbol{F}$.

Lemma 2. *Each of the* 1*-factorization* **F** *on the set* P_1 *and the* 1*-factorization* **G** *on the set* P_2 *has exactly* 12 *sub-*1*-factorizations of* $\mathbf{K}_4$.

Proof. The proof depends on the fact that if there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$, then $e \in \{x, y, z, w\}$.

Moreover, the sub-1-facorizations of $\mathbf{K}_4$ in both $\boldsymbol{F}$ and $\boldsymbol{G}$ are determined by:

$\boldsymbol{f} = \{f_i = \{e\ a_i, a_j\ a_k\}, f_j = \{e\ a_j, a_i\ a_k\}, f_k = \{e\ a_k, a_i\ a_j\}\}$ and

$\boldsymbol{g} = \{g_i = \{b\ b_i, b_j\ b_k\}, g_j = \{b\ b_j, b_i\ b_k\}, g_k = \{b\ b_k, b_i\ b_j\}\}$ for all $\{i, j, k\} \in X$.

Accordingly, we may easily verify the following two lemmas.

Lemma 3: *Let* $C_1 = \{e, a_i, a_j, a_k\}$ *be a subsloop of* $\mathbf{L}_1$. *Then* $2 \otimes_{\alpha_1} \boldsymbol{C_1} = (C_1 \cup C_2; \cdot , e)$ *is a subsloop of* $2 \otimes_\alpha \boldsymbol{L_1}$ *if and only if* $\{\alpha(i), \alpha(j), \alpha(k)\}$ *is a line in X. Where* α_1 *is equal to* α *restricted on the line* $\{i, j, k\}$ *and* $C_2 = \{b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$.

The next lemma shows that the converse of the above lemma is also true.

Lemma 4. *Let* $\boldsymbol{S} = (S; \cdot , e)$ *be a subsloop of cardinality* 8 *of* $2 \otimes_\alpha \boldsymbol{L_1}$, *then there is a* 4-*element subsloop* $C_1 = \{e, a_i, a_j, a_k\}$ *of* $\boldsymbol{L_1}$ *and a 4-element subset* $C_2 = \{b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ *of* P_2 *satisfying* $\boldsymbol{S} = 2 \otimes_{\alpha_1} \boldsymbol{C_1} = (C_1 \cup C_2; \cdot , e)$ *such that* $\{\alpha(i), \alpha(j), \alpha(k)\}$ *is a line of X. Where* α_1 *is equal to the permutation* α *restricted on the subset* $\{i, j, k\}$ *and the binary operation* " $\cdot$ " *is the same binary operation defined on* $2 \otimes_\alpha \boldsymbol{L_1}$.

Accordingly, we may say that the only possible nontrivial subsloops of $2 \otimes_\alpha \boldsymbol{L_1}$ are $\boldsymbol{L_1}$ (exactly one sub-**SL**(10)) and n $(0 \le n \le 12)$ subsloops of cardinality 8. The intersection between $\boldsymbol{L_1}$ and each of the 8-element subsloops is a 4-element subsloop. Which implies that any proper subsloop $\boldsymbol{S}$ of $2 \otimes_\alpha \boldsymbol{L_1}$ with $\boldsymbol{S} \neq \boldsymbol{L_1}$ may be determined by the set of elements $S = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ such that $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\} \in X$.

3. Subdirectly irreducible sloops of cardinality 20

Any sloop of cardinality 20 has at most one subsloop of cardinality 10 and at most 12 subsloops of cardinality 8. In particular, the direct product sloop **SL**(10) × **SL**(2) has exactly one sub-**SL**(10) and 12 sub-**SL**(8)s, but the planar **SL**(20) has no nontrivial subsloops.

In the next theorem we exhibit all nonsimple subdirectly irreducible **SL**(20)s having a sub-**SL**(10).

Theorem 5. *The constructed sloop* $2 \otimes_\alpha \boldsymbol{L_1} = (P = P_1 \cup P_2; \cdot , e)$ *is isomorphic to the direct product of the subsloop* **SL**(10) = $\boldsymbol{L_1}$ *and the* 2-*element sloop* **SL**(2), *if and only if* $2 \otimes_\alpha \boldsymbol{L_1}$ *has* 12 *sub*-**SL**(8)s, *otherwise* $2 \otimes_\alpha \boldsymbol{L_1}$ *is nonsimple subdirectly irreducible. Moreover, the constructed sloop* $2 \otimes_\alpha \boldsymbol{L_1}$ *has exactly n subsloops of cardinality* 8 *if and only if the* *permutation* α *transfers n lines into n lines of X for n* = 0, 1, 2, 3, 4, 6, 12. *Where X is the set of lines of the affine plane over* **GF**(3).

Proof. Let $2 \otimes_\alpha \boldsymbol{L_1}$ have 12 sub-**SL**(8)s, then $\alpha(X) := \{\{\alpha(i), \alpha(j), \alpha(k)\} : \text{for all } \{i, j, k\} \in X\} = X$. Consider the map φ from $2 \otimes_\alpha \boldsymbol{L_1}$ to the direct product $\boldsymbol{L_1} \times \{0, 1\}$ by $\varphi(e) = (1, 0)$, $\varphi(b) = (1, 1)$, $\varphi(a_i) = (a_i, 0)$ and $\varphi(b_i) = (a_{\alpha^{-1}(i)}, 1)$. It is routine matter to proof that φ is an isomorphism. Notice that $\varphi(a_i\, b_j) = \varphi(b_k)$ if $b_j\, b_k \in G_{\alpha(i)}$, so $\{\alpha(i), j, k\}$ is a line in X. Also, $\varphi(b_k) = (a_{\alpha^{-1}(k)}, 1)$ and $\varphi(a_i)\, \varphi(b_j) = (a_i, 0)\, (a_{\alpha^{-1}(j)}, 1) = (a_i\, a_{\alpha^{-1}(j)}, 1)$, but $\alpha^{-1}\{\alpha(i), j, k\} = \{i, \alpha^{-1}(j), \alpha^{-1}(k)\}$ is also a line in X, so $\varphi(a_i)\, \varphi(b_j) = \varphi(b_k)$.

Since $2 \otimes_\alpha \boldsymbol{L_1}$ has exactly one normal subsloop $\boldsymbol{L_1}$ of cardinality 10, so $2 \otimes_\alpha \boldsymbol{L_1}$ is not simple. Another possible normal subsloop is the 2-element subsloop $\boldsymbol{C}_2$ with $\boldsymbol{C}_2 \cap \boldsymbol{L_1} = \{e\}$. But if $2 \otimes_\alpha \boldsymbol{L_1}$ contains $\boldsymbol{C}_2$ as a normal subsloop, then $2 \otimes_\alpha \boldsymbol{L_1}$ is isomorphic to the direct product **SL**(10) × **SL**(2) and has exactly 12 sub-**SL**(8)s. Therefore, if $2 \otimes_\alpha \boldsymbol{L_1}$ has n sub-**SL**(8)s with $n < 12$, then the congruence lattice of $2 \otimes_\alpha \boldsymbol{L_1}$ has only the normal subsloop $\boldsymbol{L_1}$. Hence $2 \otimes_\alpha \boldsymbol{L_1}$ is subdirecly irreducible for all possible $n < 12$.

Let α transfer the line $\{i, j, k\} \in X$ into the line $\{\alpha(i), \alpha(j), \alpha(k)\} \in X$. According to Lemmas 3 and 4, we may directly say that $S = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ forms a subsloop. Since α is a permutation on the set of points $N = \{1, 2, \ldots, 9\}$ of the affine plane over **GF**(3), the possible values of the number n of lines of X transferred into lines are 0, 1, 2, 3, 4, 6 or 12. This completes the proof of the theorem.

In fact, there is another class of subdirectly irreducible **SL**(20)s having exactly one proper normal sub-**SL**(2) but no sub-**SL**(10). It will be described in the next section.

In [1] the author has given a construction of subdirecly irreducible sloops of cardinality 2 m. This construction supplies us with an example of a subdirectly irreducible **SL**(20) of one of these classes.

According to the results given in [12], the variety $\mathbf{V}_1$ generated by the **SL**(10) = $\boldsymbol{L}_1$ covers the smallest nontrivial subvariety $\mathbf{V}_0$ (the class of all Boolean sloops). The constructed subdirectly irreducible sloop $2 \otimes_\alpha \boldsymbol{L}_1$ = **SL**(20) contains always a subsloop of cardinality 10 and has only one proper homomorphic image isomorphic to the Boolean sloop **SL**(2). According to the result given [1], we may deduce that each of these sloops **SL**(20) = $2 \otimes_\alpha \boldsymbol{L_1}$ generates a variety $\mathbf{V}_2$ covers the variety $\mathbf{V}_1$.

4. Semi-planar sloops of cardinality 20

A semi-planar sloop is a simple sloop each of whose triangles generates either the whole sloop or a sub-**SL**(8) " cf. [2, 15]". The associated **STS**s with the semi-planar sloops will also be called semi-planar (or more precisely semi-7-planar). Each semi-planar **SL**(20) contains sub-**SL**(8)s but no sub-**SL**(10)s. Based on the number n of sub-**SL**(8)s of **SL**(20), we will determine all classes of simple **SL**(20)s. So we have six distinct classes of simple sloops, one of them is the class of planar **SL**(20) and the other five classes are semi-planar **SL**(20)s. In [2] the author has given another construction of a semi-planar sloop **SL**($2m$). This construction supplies us with exactly one class among these five classes.

We will modify the construction of the subdirectly irreducible **SL**(20) = $2 \otimes_\alpha \boldsymbol{L_1} = (P = P_1 \cup P_2; \cdot , e)$ to a construction of semi-planar sloop denoted by $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$. Let the associated **STS**(19) of the constructed subdirectly irreducible **SL**(20) = $2 \otimes_\alpha \boldsymbol{L_1}$ has a sub-**STS**(7) on the set of elements $A^* = \{a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$; i.e., $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines in X. We will interchange the set of blocks:

$$H = \{\{a_i, a_j, a_k\}, \{a_i, b_{\alpha(j)}, b_{\alpha(k)}\}, \{a_j, b_{\alpha(i)}, b_{\alpha(k)}\}, \{a_k, b_{\alpha(i)}, b_{\alpha(j)}\}\}$$

with the set of triples

$$R = \{\{ b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)} \}, \{ b_{\alpha(i)}, a_j, a_k\}, \{ b_{\alpha(j)}, a_i, a_k\}, \{ b_{\alpha(k)}, a_i, a_j\}\}$$

to get again an **STS**(19) = $(P^* = P^*_1 \cup P_2; B- H \cup R)$ denoted by $\underline{2} \otimes_\alpha \underline{\boldsymbol{T_1}}$. The associated sloop will be denoted by $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}} = (P = P_1 \cup P_2; \underline{\cdot} , e)$. Notice that the difference between the binary operations " $\underline{\cdot}$ " and " $\cdot$ " is only restricted on the subset of elements of A^*; i. e., $x \,\underline{\cdot}\, y = x \cdot y$ for all $x, y \in P - A^*$.

The next lemma is one of the main results of this section. to show that the new construction $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is a semi-planar sloop such that α transfers at least one line into a line and at most 6 lines into 6 lines of the affine plane over **GF**(3).

Theorem 6. *The constructed sloop* $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}} = (P = P_1 \cup P_2; \underline{\cdot} , e)$ *has no sub*-**SL**(10)*. Also*, $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ *is a semi-planar sloop having n sub*-**SL**(8)*s for each n* = 1, 2, 3, 4 *or* 6.. *Where n is the number of lines of the affine plane over* **GF**(3) *transferred into lines by the permutation* α.

Proof. Let $S = \{ x, y, z\}$ be a triangle in $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$. At first, we want to prove that the subsloop $\underline{<S>}$ in $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is equal to the whole sloop $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ or a sub-**SL**(8).

Assume that $|\underline{<S>} \cap A| \leq 2$, where $A = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, then the subsloop $<S>$ in the sloop $2 \otimes_\alpha \boldsymbol{L_1}$ is the same as the subsloop $\underline{<S>}$ in $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$. So if $\underline{<S>}$ is a sub-**SL**(10), then $\underline{<S>} = <S> = \boldsymbol{L_1}$ contradicting the fact that $a_i \,\underline{\cdot}\, a_j = b_{\alpha(k)}$ in $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$.

Also, if $\underline{<S>} \cap A = \{e, b, a_i, b_{\alpha(i)},\}, \{e, b, a_j, b_{\alpha(j)}\}$ or $\{e, b, a_k, b_{\alpha(k)}\}$, then the subsloop $<S>$ in the sloop $2 \otimes_\alpha \boldsymbol{L_1}$ is the same as the subsloop $\underline{<S>}$ in $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$. For the same reason, if $<S>$ is a sub-**SL**(10), then $\underline{<S>} = <S> = \boldsymbol{L_1}$ contradicting the fact that $b \in \underline{<S>}$.

Moreover if $|\underline{<S>} \cap A| > 4$, then $\underline{<S>} = A$; i.e., $\underline{<S>}$ is a sub-**SL**(8).

Now Assume that $|\underline{<S>}| = 10$ and

$\underline{<S>} \cap A = \{e, b_{\alpha(i)}, a_j, a_k\}, \{e, b_{\alpha(j)}, a_i, a_k\}, \{e, b_{\alpha(k)} , a_i, a_j\}$ or $\{e, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$.

Each of these four blocks contains at least one element $b_{\alpha(t)}$ lying in P_2, for $t = i, j$ or k. If b_r or $a_s \in \underline{\langle S\rangle} - A$, then $b_{\alpha(t)} \cdot b_r \in P_1$ and $b_{\alpha(t)} \cdot a_s \in P_2$, This means that the 6-element subset $\underline{\langle S\rangle} - A$ consists of two 3-element subsets $\{a_{s_1}, a_{s_2}, a_{s_3}\} \subseteq P^*_1$ and $\{b_{r_1}, b_{r_2}, b_{r_3}\} \subseteq P_2$. For the three case $\underline{\langle S\rangle} \cap A = \{e, b_{\alpha(i)}, a_j, a_k\}$, $\{e, b_{\alpha(j)}, a_i, a_k\}$ or $\{e, b_{\alpha(k)}, a_i, a_j\}$, we have $a_t \cdot \{a_{s_1}, a_{s_2}, a_{s_3}\} \neq \{a_{s_1}, a_{s_2}, a_{s_3}\}$ and $a_t \cdot \{a_{s_1}, a_{s_2}, a_{s_3}\} \cap \{e, a_i, a_j, a_k\} = \emptyset$ for $t = i, j$ or k, this means that $\underline{\langle S\rangle}$ consists of a 4-element subset of P_2 and more than 6 elements lying in L_1, hence $\underline{\langle S\rangle}$ has more than 10 elements, so $\underline{\langle S\rangle}$ must be equal to $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$.
For the case $\underline{\langle S\rangle} \cap A = \{e, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, the set $\underline{\langle S\rangle} - A$ contains $\{a_{s_1}, a_{s_2}, a_{s_3}\} \subseteq P^*_1$ and $\{b_{r_1}, b_{r_2}, b_{r_3}\} \subseteq P_2$. Let $\underline{\langle S\rangle}^*$ be the associated **STS**(9) with $\underline{\langle S\rangle}$, Since $\{a_{s_1}, a_{s_2}, a_{s_3}\} \cap \{a_i, a_j, a_k\} = \emptyset$, the triple $\{a_{s_1}, a_{s_2}, a_{s_3}\}$ forms a block of $\underline{\langle S\rangle}^*$. If $\underline{\langle S\rangle}$ is a sub-**SL**(10), then the triple $\{b_{r_1}, b_{r_2}, b_{r_3}\}$ must also be a block of $\underline{\langle S\rangle}^*$ contradicting the fact that the construction $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ contains exactly one block lying completely in P_2 that is the block $\{b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\} \subseteq P_2$. Therefore, the subsloop $\underline{\langle S\rangle}$ generated by any triangle S is equal to a sub-**SL**(8) or the whole sloop $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$. This means that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has no sub-**SL**(10)s for all n = 1, 2, 3, 4, 6 or 12, .
Secondly, we have to prove that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has no proper congruence for n = 1, 2, 3, 4 and 6. Assume that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has a congruence θ with $[e]\theta = \{e, x\}$. If $[e]\theta \cap A = \{e\}$, then $[A]\theta$ is a sub-**SL**(16) which is impossible. Hence $[e]\theta \cap A = [e]\theta$. Say $[e]\theta = \{e, a_i\} \subseteq A$ and suppose that $\{a_j, a_r, a_s\}$ is a block such that $\{a_j, a_r, a_s\} \cap \{a_i, a_j, a_k\} = \{a_j\}$ for $i \neq j$, so we have $[e]\theta \cup [a_j]\theta \cup [a_r]\theta \cup [a_s]\theta = \{e, a_i, a_j, a_i \cdot a_j, a_r, a_i \cdot a_r, a_s, a_i \cdot a_s\} = \{e, a_i, a_j, b_{\alpha(k)}, a_r, a_l, a_s, a_h\}$, where $a_l = a_i \cdot a_r$ and $a_h = a_i \cdot a_s$. But $b_{\alpha(k)} \cdot a_r = b_v \neq b_{\alpha(k)}$ contradicting that $[e]\theta \cup [a_j]\theta \cup [a_r]\theta \cup [a_s]\theta$ is an 8-element subsloop.
Now assume that $[e]\theta = \{e, b_{\alpha(i)}\} \subseteq A$ and suppose that $\{a_j, b_r, b_s\}$ is a block such that $\{a_j, b_r, b_s\} \cap A = \{a_j\}$ for $i \neq j$. So we have $[e]\theta \cup [a_j]\theta \cup [b_r]\theta \cup [b_s]\theta = \{e, b_{\alpha(i)}, a_j, b_{\alpha(i)} \cdot a_j, b_r, b_{\alpha(i)} \cdot b_r, b_s, b_{\alpha(i)} \cdot b_s\} = \{e, b_{\alpha(i)}, a_j, a_k, b_r, a_l, b_s, a_h\}$, where $b_{\alpha(i)} = a_j \cdot a_k$, $a_l = b_{\alpha(i)} \cdot b_r$ and $a_h = b_{\alpha(i)} \cdot b_s$. If $a_j \cdot a_l = a_h$, then $a_k \cdot a_l \notin [e]\theta \cup [a_i]\theta \cup [b_r]\theta \cup [b_s]\theta$ contradicting that $[e]\theta \cup [a_i]\theta \cup [b_r]\theta \cup [b_s]\theta$ must be an 8-element subsloop.
Now, assume that $[e]\theta = \{e, b\} \subseteq A$ and suppose that $\{l, m, n\}$ is a line in X such that $\{\alpha(l), \alpha(m), \alpha(n)\}$ is not a line in X, then $[e]\theta \cup [a_l]\theta \cup [a_m]\theta \cup [a_n]\theta = \{e, b, a_l, b_{\alpha(l)}, a_m, b_{\alpha(m)}, a_n, b_{\alpha(n)}\}$. But according to Lemma 4, the set $\{e, b, a_l, b_{\alpha(l)}, a_m, b_{\alpha(m)}, a_n, b_{\alpha(n)}\}$ does not form an **SL**(8). This means that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has no congruence θ with $[e]\theta = \{e, x\}$, which implies that the constructed $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is a semi-planar **SL**(20) for all n = 1, 2, 3, 4, 6. Therefore, the proof of the theorem is complete.

Finally, for n =12, the permutation α transfers each line of X into a line. Then the constructed sloop $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has 12 **SL**(8)s but no **SL**(10)s. It will be shown that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is not simple. The next theorem shows that $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ in this case is subdiretly irreducible having exactly one normal subsloop that is a sub-**SL**(2).

Theorem 7. *Let* $\boldsymbol{L}$ *be a sloop of cardinality* $2m$. *A subsloop* $S = \{e, x\}$ *is normal if and only if* $\boldsymbol{L}$ *contains* $(m - 1)(2m - 4)/12$ *sub-*$\mathbf{SL}(8)$*s including the element* x.

Proof. If $\boldsymbol{S}$ is a normal subsloop of $\boldsymbol{L}$, then $\boldsymbol{L/S}$ is an $\mathbf{SL}(m)$. An $\mathbf{SL}(m)$ has $(m - 1)(m - 2)/6$ 4-element subsloops, which implies that $\boldsymbol{L}$ has $(m - 1)(m - 2)/6$ sub-$\mathbf{SL}(8)$s containing $\boldsymbol{S}$.
On the another direction, the number of triangles of an $\mathbf{SL}(2m)$ passing through a fixed point x is equal to $(2m - 2)(2m - 3)/2 - (2m - 2)/2 = (m - 1)(2m - 4)$ and the number of triangles of an $\mathbf{SL}(8)$ passing through the fixed point x is equal to 12, then the maximum number of sub-$\mathbf{SL}(8)$s of an $\mathbf{SL}(20)$ containing x is equal to $(m - 1)(2m - 4)/12$. This means that if $\boldsymbol{L}$ contains $(m - 1)(2m - 4)/12$ sub-$\mathbf{SL}(8)$s passing through x, then each triangle in $\boldsymbol{L}$ generates a sub-$\mathbf{SL}(8)$. Let $y, z \in L - \{e, x\}$. If $\{e, x, y, z\}$ forms a sub-$\mathbf{SL}(4)$ then $y \cdot z \cdot (\boldsymbol{S}) = y \cdot (z \cdot (\boldsymbol{S}))$. If $\{e, x, y, z\}$ does not form a sub-$\mathbf{SL}(4)$, then $\{x, y, z\}$ is a triangle in $\boldsymbol{L}$ and the subsloop generated by $\{x, y, z\}$ is an $\mathbf{SL}(8)$. It is well known that an $\mathbf{SL}(8)$ is always Boolean. This implies that $y \cdot z \cdot (\boldsymbol{S}) = y \cdot (z \cdot (\boldsymbol{S}))$. So $\boldsymbol{S}$ is normal. This completes the proof of the lemma.

According to the above theorem, if the constructed $\mathbf{SL}(2m) = 2 \otimes_\alpha \boldsymbol{L_1}$ has a simple $\mathbf{SL}(m) = \boldsymbol{L_1}$ and $(m - 1)(m - 2)/6$ sub-$\mathbf{SL}(8)$s passing through sub-$\mathbf{SL}(2) = \{e, x\}$, so $\{e, x\}$ is normal. Since $\boldsymbol{L_1}$ is simple then $L_1 \cap \{e, x\} = \{e\}$. According to the definition of the constructed $2 \otimes_\alpha \boldsymbol{L_1}$, we have $x = b$, so the subsloops $\boldsymbol{L_1}$ and $\{e, b\}$ are normal, then $2 \otimes_\alpha \boldsymbol{L_1}$ is isomorphic to the direct product $\mathbf{SL}(m) \times \mathbf{SL}(2)$. This result agrees with result of Theorem 5.

Also, for $m = 10$ in the above theorem and according to Theorem 6, we may say that for $n = 12$ the constructed sloop $\mathbf{SL}(20) = \underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has $(10 - 1)(20 - 4)/12 = 12$ sub-$\mathbf{SL}(8)$s passing through sub-$\mathbf{SL}(2) = \{e, b\}$, but no sub-$\mathbf{SL}(10)$. So $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ has exactly one proper congruence θ with $[e]\theta = \{e, b\}$. This means that the constructed sloop $\mathbf{SL}(20) = \underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is subdirectly irreducible having only one proper homomorphic image isomorphic to $\mathbf{SL}(10)$.

According to the results due to Quackenbush in [12], the variety $\mathbf{V}_1$ generated by $\mathbf{SL}(10)$ covers the smallest nontrivial subvariety $\mathbf{V}_0$ (the class of all Boolean sloops). And according to [2], we may deduce that each of the constructed semi-planar sloop $\mathbf{SL}(20) = \underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ generates also a variety $\mathbf{V}'_1$ (not comparable with $\mathbf{V}_1$) covering the variety of all Boolean sloops $\mathbf{V}_0$.

5. Construction an example of each class of SL(20)s

Let $(P^*_1; B_1)$ be an $\mathbf{STS}(9)$, where $P^*_1 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ and the set of blocks B_1 is given by:

$B_1 = \quad a_1\, a_3\, a_4 \qquad a_1\, a_2\, a_6 \qquad a_1\, a_7\, a_8 \qquad a_1\, a_5\, a_9 \qquad a_2\, a_3\, a_9 \qquad a_2\, a_4\, a_7$

$a_2\, a_5\, a_8$ $a_3\, a_5\, a_7$ $a_3\, a_6\, a_8$ $a_4\, a_5\, a_6$ $a_4\, a_8\, a_9$ $a_6\, a_7\, a_9$

Let $\boldsymbol{L}_1 = (P_1 = P^*_1 \cup \{e\}; \cdot , e)$ be the associated sloop of $(P^*_1; B_1)$. Also, let $P_2 = \{b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\}$. The 1-factorization $\boldsymbol{F}$ on the set P_1 and the 1-factorization $\boldsymbol{G}$ on the set P_2 are defined as in section 2 by:

$\boldsymbol{F} = \{F_1, F_2, ... , F_9\}$, where $F_i = \{a_l\, a_k : a_l \,.\, a_k = a_i$ and $a_i, a_l, a_k \in P_1\}$ and

$\boldsymbol{G} = \{G_1, G_2, ... , G_9\}$, where $G_i = \{b\, b_i\} \cup \{b_l\, b_k : a_l \cdot a_k = a_i$ for $i \notin \{l, k\}\}$.

The constructed **STS**(19) = $2 \otimes_\alpha \boldsymbol{T_1}$ is defined by $(P^* = P^*_1 \cup P_2; B = B_1 \cup B_{12})$, where $B_{12} = \{\{a_i, b_j, b_k\} : b_j\, b_k \in G_{\alpha(i)}\}$. The associated sloop $2 \otimes_\alpha \boldsymbol{L_1} = (P = P^* \cup \{e\}; \cdot , e)$ with the **STS**(19) = $2 \otimes_\alpha \boldsymbol{T_1}$ has the sub-**SL**(10) = $\boldsymbol{L_1}$ for each permutation α, so $\boldsymbol{L_1}$ is always normal in $2 \otimes_\alpha \boldsymbol{L_1}$.

For each block $\{a_i , a_j , a_k\} \in B_1$, we have the sub-1-factorizations:

$\boldsymbol{f} = \{f_i = \{e\, a_i, a_j\, a_k\}, f_j = \{e\, a_j, a_i\, a_k\}, f_k = \{e\, a_k, a_i\, a_j\}\}$ and

$\boldsymbol{g} = \{g_i = \{b\, b_i, b_j\, b_k\}, g_j = \{b\, b_j, b_i\, b_k\}, g_k = \{b\, b_k, b_i\, b_j\}\}$ for all $\{i, j, k\} \in X$.

Where $X = \{\{1, 3, 4\}, \{1, 2, 6\}, \{1, 7, 8\}, \{1, 5, 9\}, \{2, 3, 9\}, \{2, 4, 7\}, \{2, 5, 8\}, \{3, 5, 7\}, \{3, 6, 8\}, \{4, 5, 6\}, \{4, 8, 9\}, \{6, 7, 9\}\}$ is the set of lines of the affine planar over **GF**(3) with the set of points $N = \{1, 2, \dots , 9\}$.

By applying the interchange:

$H = \{\{a_1, a_3, a_4\}, \{a_1, b_3, b_4\}, \{a_3, b_1, b_4\}, \{a_4, b_1, b_3\}\}$

with the set of triples:

$R = \{\{b_1, b_3, b_4\}, \{b_1, a_3, a_4\}, \{b_3, a_1, a_4\}, \{b_4, a_1, a_3\}\}$

on the set $A^* = \{a_1, a_3, a_4, b, b_1, b_3, b_4\}$, we will get the associated sloop **SL**(20) = $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ with the constructed triple system $\underline{2} \otimes_\alpha \underline{\boldsymbol{T_1}} = (P^*; B - H \cup R)$.

The following 7 examples supplies us with an example for each class of **SL**(20) given in section 3 and in section 4.

Notice that {1, 3, 4} is a line in *X*. We will choose the permutation α satisfying that α(1) = 1, α(3) = 3 and α(4) = 4 in all examples from (1) to (6):

(1) $\alpha_1 = id_N$; i. e., α_1 transfers each line into the same line in *X*. The constructed **SL**(20) = $2 \otimes_{\alpha_1} \boldsymbol{L_1}$ has 12 sub-**SL**(8)s and one sub-**SL**(10); i. e. $2 \otimes_{\alpha_1} \boldsymbol{L_1}$ is isomorphic to **SL**(10) × **SL**(2). And the constructed $\underline{2} \otimes_{\alpha_1} \underline{\boldsymbol{L_1}}$ is an **SL**(20) having 12 sub-**SL**(8)s but no sub-**SL**(10). The sloop $\underline{2} \otimes_{\alpha_1} \underline{\boldsymbol{L_1}}$ is subdirectly irreducible having exactly one proper homomorphic image ≅ **SL**(10).

In all cases (2) – (7), the constructed **SL**(20) = $2 \otimes_\alpha \boldsymbol{L_1}$ is subdirectly irreducible. And in all cases (2) - (6), the constructed **SL**(20) = $\underline{2} \otimes_\alpha \underline{\boldsymbol{L_1}}$ is a semi-planar sloop.

(2) $\alpha_2 = (26)$; i. e., α_2 transfers 6 lines into lines, namely, the set {{1, 3, 4}, {1, 2, 6}, {1, 7, 8}, {1, 5, 9}, {4, 8, 9}, {3, 5, 7}}. The constructed **SL**(20) = $2 \otimes_{\alpha_2} \boldsymbol{L_1}$ has one sub-**SL**(10) and 6 sub-**SL**(8)s. The constructed **SL**(20) = $\underline{2} \otimes_{\alpha_2} \underline{\boldsymbol{L_1}}$ has 6 sub-**SL**(8)s but no sub-**SL**(10).

(3) $\alpha_3 = (26)(78)$; i. e., α_3 transfers 4 lines into lines, where the set of the four lines is {{1, 3, 4}, {1, 2, 6}, {1, 7, 8}, {1, 5, 9}}. The constructed **SL**(20) = $2 \otimes_{\alpha_3} \boldsymbol{L_1}$ has one sub-**SL**(10) and 4 sub-**SL**(8)s. The constructed **SL**(20) = $\underline{2} \otimes_{\alpha_3} \underline{\boldsymbol{L_1}}$ has 4 sub-**SL**(8)s but no sub-**SL**(10).

(4) $\alpha_4 = (258)$; i. e., α_4 transfers 3 lines into lines, where the set of the three lines is {{1, 3, 4}, {2, 5, 8}, {6, 7, 9}}. The constructed **SL**(20) = $2 \otimes_{\alpha_4} \boldsymbol{L_1}$ has one sub-**SL**(10) and 3 sub-**SL**(8)s. The constructed **SL**(20) = $\underline{2} \otimes_{\alpha_4} \underline{\boldsymbol{L_1}}$ has 3 sub-**SL**(8)s but no sub-**SL**(10).

(5) $\alpha_5 = (2567)(89)$; i. e., α_5 transfers 2 lines into lines, where the set of lines is {{1, 3, 4}, {4, 8, 9}}. The constructed **SL**(20) = $2 \otimes_{\alpha_5} \boldsymbol{L_1}$ has one sub-**SL**(10) and 2 sub-**SL**(8)s. The constructed **SL**(20) = $\underline{2} \otimes_{\alpha_5} \underline{\boldsymbol{L_1}}$ has 2 sub-**SL**(8)s but no sub-**SL**(10).

(6) $\alpha_6 = (257968)$; i. e., α_6 transfers only the line {1, 3, 4} into a line. The constructed **SL**(20) = $2 \otimes_{\alpha_6} \boldsymbol{L_1}$ has only one sub-**SL**(10) and one sub-**SL**(8). The constructed **SL**(20) = $\underline{2} \otimes_{\alpha_6} \underline{\boldsymbol{L_1}}$ has only one sub-**SL**(8) but no sub-**SL**(10).

(7) $\alpha_7 = (123456798)$; i. e., α_7 transfers no line into a line. The constructed **SL**(20) = $2 \otimes_{\alpha_7} \boldsymbol{L_1}$ has only one sub-**SL**(10) and no sub-**SL**(8)s. In fact, the corresponding **STS**(19) = $2 \otimes_{\alpha_7} \boldsymbol{T_1}$ has exactly one sub-**STS**(9), but no sub-**STS**(7)s. This means that each triangle in the associated squag **SQ**(19) either generates the whole **SQ**(19) or a sub-**SQ**(9). Which implies that the associated squag **SQ**(19) is an example of a semi-planar squag of cardinality 19. We note that the smallest known cardinality of semi-planar squag is 21 (cf. [2]).

The subsloops mentioned in the above examples are **SL**(10) = $\boldsymbol{L}_1$, in which $\boldsymbol{L}_1$ is always normal in $2 \otimes_\alpha \boldsymbol{L_1}$ and the sub-**SL**(8)s determined by the set $\{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, in which $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines of the affine plane over **GF**(3).

References

[1] **M.H. Armanious**, *On Subdirectly Irreducible Steiner Loops of Cardinality* $2n$, Beitrage zur Algebra und Geometrie, Vol. 43 (2002), No. 2, 325-331.

[2] **M.H. Armanious**, *Semi-planar Steiner Loops of Cardinality* $2n$, Discrete Math. 270 (2003) 291-298.

[3] **M.H. Armanious**, *Semi-planar Steiner Quasigroups of Cardinality* $3n$, Australas. J. Combin., 27 (2003), 13-21.

[4] **M. H. Armanious**, **E. M. A. Elzayat**, *Extending Sloops of Cardinality 16 to SQS-Skeins with all Possible Congruence Lattices*, Quasigroups and Related Systems, Vol. 12 (2004), 1-12.

[5] **O. Chein, H.O. Pflugfelder** and **J. D. H. Smith**, *Quasigroups and Loops*, *Theory and Applications*, Sigma Series in Pure Math., 8, Heldermann Verlag, Berlin, 1990.

[6] **C. Colburn** and **J. Dinitz**, eds., *The CRC Handbook of Combinatorial Designs*, CRC Press, New York, 1996.

[7] **J. Doyen**, *Sur la Sttucture de Certains Systems Triples de Steiner*, Math. Z. 111 (1979) 289-300.

[8] **B. Ganter**, and **H. Werner**, *Co-ordinatizing Steiner Systems*, Ann. Discrete Math. 7 (1980) 3- 24.

[9] **G. Grätzer**, *Universal Algebra*, Springer – Verlag New York, Heidelberg and Berlin, 2nd edition, 1997.

[10] **F. Haray**, *Graph Theory*, Addison-Wesley, Reading, MA (1969).

[11] **C. C. Lindner**, and **A. Rosa**, *Steiner Quadruple Systems*: a Survey Discrete Math.21 (1979) 147-181.

[12] **R.W. Quackenbush**, *Varieties of Steiner Loops and Steiner Quasigroups*, Can. J. Math. (1976) 1187-1198.

Appendix including proofs of lemmas 1, 2, 3 and 4

Proof of lemma 1. We have that $P_1:=\{e, a_1, \dots, a_n\}$ and $P_2 :=\{b, b_1, \dots, b_n\}$ are disjoint sets having the same cardinality n, in which P_1 forms a subsloop of $2 \otimes_\alpha \boldsymbol{L_1}$. Let $\boldsymbol{S}$ satisfy $S - P_i \neq\varnothing$ for $i = 1$ and 2, then $|S| \geq 2$. If $|S| = 2$, then $S = \{a_i, b_j\}$. For $|S| > 2$, since $a_i \cdot b_k$ are always element of P_2, for all $a_i \in P_1$ and $b_k \in P_2$, then S contains at least a 2-element subset $\{a_i, a_j\} \subseteq P_1$ and a 2-element subset $\{b_k, b_l\} \subseteq P_2$. Consider the map $\delta_{b_k}(x) := x \cdot b_k$ for $x \in S \cap P_1$. It is easy to see that the map δ_{b_j} is bijective from the subset $S \cap P_1$ onto the subset $S - P_1$. Which implies that $|S \cap P_1| = (1/2)|S| = |S \cap P_2|$.

Proof of lemma 2. Let $\boldsymbol{f}$ be a sub-1-factorization on $\mathbf{K}_r$ of $\boldsymbol{F}$. Then the order r of the complete graph $\mathbf{K}_r$ is an even number less than or equal 10/2 = 5, hence $r = 4$. Indeed, if there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$, then $e \in\{x, y, z, w\}$. Otherwise, assume that $e \notin \{x, y, z, w\}$ and $\{x\, y, z\, w\} \subseteq F_i$, $\{x\, z, y\, w\} \subseteq F_j$ and $\{x\, w, y\, z\} \subseteq F_k$ form a sub-1-factorization of $\mathbf{K}_4$. But $e\, a_i \in F_i$, $e\, a_j \in F_j$ and $e\, a_k \in F_k$, then $\{e, a_i, a_j, a_k\} \cap \{x, y, z, w\} = \varnothing$. This implies that $F_i = \{e\, a_i, a_j\, a_k, x\, y, z\, w, u\, v\}$ or $\{e\, a_i, a_j\, u, a_k\, v, x\, y, z\, w\}$, Hence the first case of F_i tends to the 1-factor $F_j =\{e\, a_j, a_i\, a_k, x\, z, y\, w, u\, v\}$ and the second case of F_i leds to the 1-factor $F_j = \{e\, a_j, a_i\, u, a_k\, v, x\, z, y\, w\}$, both cases contradict the fact that $F_i \cap F_j = \varnothing$. Similarly, if there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$ of P_2, then b must be an element of $\{x, y, z, w\}$. Since the number of blocks of B_1 is 12, each of the 1-factorizations $\boldsymbol{F}$ and $\boldsymbol{G}$ has exactly 12 sub-1-facorizations of $\mathbf{K}_4$. This completes the proof.

Proof of lemma 3. Let $2 \otimes_{\alpha_1} \boldsymbol{C_1} = (C_1 \cup C_2\,; \cdot\,, e)$ be a subsloop of $2 \otimes_\alpha \mathbf{L}_1$. Since $C_1 = \{e, a_i, a_j, a_k\}$ is a subsloop of $\boldsymbol{L}_1$, there is a sub-1-factorization $\boldsymbol{f} = \{f_i, f_j, f_k\}$ on C_1. According to the definition of $2 \otimes_{\alpha_1} \boldsymbol{C_1}$, there a sub-1-factorization $\boldsymbol{g} = \{g_{\alpha(i)} = \{b\ b_{\alpha(i)}, b_{\alpha(j)}\ b_{\alpha(k)}\} \subseteq G_{\alpha(i)}$, $g_{\alpha(j)} = \{b\ b_{\alpha(j)}, b_{\alpha(i)}\ b_{\alpha(k)}\} \subseteq G_{\alpha(j)}$, $g_{\alpha(k)} = \{b\ b_{\alpha(k)}, b_{\alpha(i)}\ b_{\alpha(j)}\} \subseteq G_{\alpha(k)}$ on the subset $C_2 = \{b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ if and only if $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line of X. This implies that $2 \otimes_{\alpha_1} \boldsymbol{C_1} = (C_1 \cup C_2; \cdot\,, e)$ is a subsloop of $2 \otimes_\alpha \boldsymbol{L_1}$, if and only if $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in X.

Proof of lemma 4. According to Lemma 2, we may say that $S \cap L_1 = C_1 = C_1 = \{e, a_i, a_j, a_k\}$ is a 4-element subsloop and $\{i, j, k\}$ is a line in X. So there is a sub-1-factorization $\boldsymbol{f} = \{f_i = \{e\ a_i, a_j\ a_k\}, f_j = \{e\ a_j, a_i\ a_k\}, f_k = \{e\ a_k, a_i\ a_j\}\}$ on C_1. According to the construction $2 \otimes_\alpha \boldsymbol{L_1}$ we have: f_i related with $G_{\alpha(i)}$, f_j related with $G_{\alpha(j)}$ and f_k related with $G_{\alpha(k)}$. Since $\boldsymbol{S}$ is a sub-**SL**(8), then$\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in X and the three 1-factors $G_{\alpha(i)}$, $G_{\alpha(j)}$ and $G_{\alpha(k)}$ contains a sub-1-factorization $\boldsymbol{g} = \{g_{\alpha(i)} = \{b\ b_{\alpha(i)}, b_{\alpha(j)}\ b_{\alpha(k)}\}, g_{\alpha(j)} = \{b\ b_{\alpha(j)}, b_{\alpha(i)}\ b_{\alpha(k)}\}, g_{\alpha(k)} = \{b\ b_{\alpha(k)}, b_{\alpha(i)}\ b_{\alpha(j)}\}\}$ on the 4-element subset $C_2 = \{b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ *of* P_2. According to the definition of the set of blocks B_{12} and using the sub-1-factorizations $\boldsymbol{f}$ and $\boldsymbol{g}$, then the subsloop $\boldsymbol{S}$ can be represented by the construction $2 \otimes_{\alpha_1} \boldsymbol{C_1} = (C_1 \cup C_2; \cdot\,, e)$, where α_1 is equal to α restricted on the subset $\{i, j, k\}$. This completes the proof.