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# GENERALIZED FIBONACCI AND LUCAS SEQUENCES WITH PASCAL-TYPE ARRAYS

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**ABSTRACT**

We re-label the Fibonacci and Lucas sequences respectively by

$$\{F_{0,n}\} \equiv \{F_n\} \text{ and } \{F_{1,n}\} \equiv \{L_n\},$$

and consider

$$F_{m,n} = F_{m-1,n-1} + F_{m-1,n+1},\ m, n \geq 1,$$

as a generalization of the well-known identity

$$L_n = F_{n-1} + F_{n+1},$$

where

$$F_{m,n} = F_{m,n-1} + F_{m,n-2},\ m \geq 1, n > 2.$$

## 1. INTRODUCTION

The purpose of this paper is primarily to collate and to relate a number of known Fibonacci results which are scattered in the literature. This is done in the context of a slightly new generalization of these numbers.

The paper is in two inter-related parts. In the first part, we generalize a result which connects the Fibonacci and Lucas numbers, namely

$$L_n = F_{n-1} + F_{n+1},\ \ n \geq 1. \tag{1.1}$$

Arising out of this is a Pascal-type array which is related to the Lucas numbers in the same way that the Pascal array is related to the Fibonacci numbers. The second part of the paper then considers powers of such arrays and their rising diagonal sums which are elements of a generalized Pell sequence defined by

$$u_n = ku_{n-1} + u_{n-2},\ n \geq 0,\ u_0 = 1, u_1 = k, \tag{1.2}$$

which yields the ordinary Fibonacci numbers when $k$=1, and the ordinary Pell numbers when $k$=2.

## 2. A GENERALIZATION OF A FIBONACCI-LUCAS IDENTITY

Consider the second order linear recursive sequences defined by the recurrence relation

$$F_{m,n} = F_{m,n-1} + F_{m,n-2}, m \geq 0, \ n > 2, \tag{2.1}$$

with initial terms $F_{2m,1} = F_{2m,2} = F_{2m+1,1} = 5^m, F_{2m+1,2} = 3 \times 5^m$, so that the Fibonacci and Lucas sequences can be re-labelled as

$$\{F_{0,n}\} \equiv \{F_n\} \text{ and } \{F_{1,n}\} \equiv \{L_n\}.$$

The sets of sequences are then inter-related by

$$F_{m,n} = F_{m-1,n-1} + F_{m-1,n+1}, \ m \geq 1, n > 2. \tag{2.2}$$

*Proof*: The result follows from induction on $n$. It is readily verified for $n$=3. Suppose the result is true for $n$=4,5,…,$k$. Then

$$\begin{aligned} F_{m,k} &= F_{m,k-1} + F_{m,k-2} && \text{(from (2.1)} \\ &= \left(F_{m-1,k-2} + F_{m-1,k}\right) + \left(F_{m-1,k-3} + F_{m-1,k-1}\right) && \text{(inductive assumption)} \\ &= \left(F_{m-1,k-2} + F_{m-1,k-3}\right) + \left(F_{m-1,k} + F_{m-1,k-1}\right) \\ &= F_{m-1,k-1} + F_{m-1,k+1}, \end{aligned}$$

as required.

Equation (2.2) is the same partial difference equation which is used with different boundary conditions to model lattice paths and to generate Catalan numbers in Carlitz and Riordan [5], while (2.1) and (2.2) are similarly used by Hosoya [12] to generate magic diamonds. Other similarities are indicated in Bondarenko [3].

It can then also be proved by induction that

$$F_{m,n} = \sum_{j=0}^{m} \binom{m}{j} F_{0,n+2j-m}. \tag{2.3}$$

For example,

$$\begin{aligned} F_{2,n} &= F_{1,n-1} + F_{1,n+1} \\ &= F_{0,n-2} + F_{0,n} + F_{0,n} + F_{0,n+2} \\ &= F_{0,n-2} + 2F_{0,n} + F_{0,n+2}, \end{aligned}$$

and

$$F_{m,n} = F_{m-1,n-1} + F_{m-1,n+1}, \ m \geq 1, n > 1. \tag{2.4}$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $F_{0,n}$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 |
| $F_{1,n}$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 |
| $F_{2,n}$ | 5 | 5 | 10 | 15 | 25 | 40 | 65 |
| $F_{3,n}$ | 5 | 15 | 20 | 35 | 55 | 90 | 145 |
| $F_{4,n}$ | 25 | 25 | 50 | 75 | 125 | 200 | 325 |

Table 1: Some examples of $\{F_{m,n}\}$

Other familiar generalizations include

$$F_{2m+1,n}F_{2m,n} = 5^{2m}F_{0,2n}$$

which reduces to

$$L_nF_n = F_{2n}$$

when $m$=0.

## 3. RISING DIAGONAL SEQUENCES

If we now consider falling diagonal sequences in Table 1, then we can define

$$U_{m,n} = F_{m-1,n+1} + F_{m-2,n},\ m > 1, n \geq 1. \tag{3.1}$$

Some examples of $U_{m,n}$ are shown in Table 2.

Note that $\{U_{2,n}\}$ was called a "conjugate sequence" by Brousseau [2] and is listed as A000285 in [16] and that $\{U_{3,n}\}$ is listed as A022388 in [16] without reference.

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $U_{2,n}$ | 4 | 5 | 9 | 14 | 23 | 37 | 60 |
| $U_{3,n}$ | 6 | 13 | 19 | 32 | 51 | 88 | 134 |
| $U_{4,n}$ | 20 | 25 | 45 | 70 | 115 | 185 | 300 |
| $U_{5,n}$ | 30 | 65 | 95 | 160 | 255 | 415 | 670 |

Table 2: Some examples of $\{U_{m,n}\}$

For example,

$$U_{k,n} = 2U_{k-1,n-1} + U_{k-1,n}, k \geq 2, n \geq 1.$$

and

$$\begin{aligned} U_{2,n} &= F_{1,n+1} + F_{0,n} \\ &= (F_{0,n} + F_{0,n+2}) + F_{0,n} \\ &= 2F_{0,n} + F_{0,n+2}, \end{aligned}$$

and

$$\begin{aligned} U_{3,n} &= F_{2,n+1} + F_{1,n} \\ &= \left(F_{1,n} + F_{1,n+2}\right) + F_{1,n} \\ &= 2F_{1,n} + F_{1,n+2} \\ &= \left(2F_{0,n-1} + 2F_{0,n+1}\right) + \left(F_{0,n+1} + F_{0,n+3}\right) \\ &= 2F_{0,n-1} + 3F_{0,n+1} + F_{0,n+3}. \end{aligned}$$

which generalizes to

$$U_{m,n} = \sum_{j=0}^{m-1} w_{m-1,j}\, F_{0,n+m-2j}, \tag{3.2}$$

where $w_{i,j}$ is defined by the partial recurrence relation (Feinberg [6])

$$w_{i,j} = w_{i-1,j} + w_{i-1,j-1}, \;\; i > 1,\, j \geq 1, \tag{3.3}$$

with boundary conditions

$$w_{i,0} = 1, i \geq 0; w_{i,i} = 2, i \geq 1; w_{i,j} = 0, i < j.$$

Proof. The proof of (3.2) follows readily with induction on *m*.

Some examples of $w_{i,j}$ are shown in Table 3, in which we note that the column sequences may be found in [1] and [16].

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 0 | 0 | 0 | 0 |
| 1 | 3 | 2 | 0 | 0 | 0 |
| 1 | 4 | 5 | 2 | 0 | 0 |
| 1 | 5 | 9 | 7 | 2 | 0 |
| 1 | 6 | 14 | 16 | 9 | 2 |

Table 3: Some examples of $\{w_{i,j}\}$

This suggests that the $\{w_{i,j}\}$ 'triangle' is related to the Pascal triangle. In fact, it is composed of the sum of two Pascal triangles, one of which is shifted down one row and across one column (Gould and Greig [9]):

$$w_{i,j} = \binom{i}{j} + \binom{i-1}{j-1}.$$

## 4. POWERS OF THE ARRAYS

We now consider these rising diagonal sequences as they arise in powers of arrays. Suppose we define the matrix *A* to the power *k* by

$$A^k = \left[a_{i,j}^{(k)}\right] \tag{4.1}$$

where

$$a_{i,j}^{(k)} = \begin{cases} 0, & \text{if } 0 \le i < j, \\ \binom{i}{j} k^{i-j}, & \text{if } i \ge j \ge 0. \end{cases} \tag{4.2}$$

Some examples of $a_{i,j}^{(k)}$ are

$$a_{4,2}^{(3)} = 54,\ a_{2,4}^{(4)} = 0,\ a_{5,1}^{(5)} = 3125,\ a_{3,2}^{(10)} = 30.$$

Examples of $A^k$ now follow with the associated rising diagonal sequences:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \quad \{u_n\} = \{1,1,2,3,5,...\},$$

(Fibonacci)

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 8 & 12 & 6 & 1 \end{bmatrix}, \quad \{u_n\} = \{1,2,5,12,29,...\},$$

(Pell)

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 9 & 6 & 1 & 0 \\ 27 & 27 & 9 & 1 \end{bmatrix}, \quad \{u_n\} = \{1,3,10,33,109,...\},$$

(Horadam [11])

$$A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 16 & 8 & 1 & 0 \\ 64 & 48 & 12 & 1 \end{bmatrix}, \quad \{u_n\} = \{1,4,17,72,305,...\}.$$

(Thébault [18])

From these it can be inferred that

$$u_n = ku_{n-1} + u_{n-2},\ n \ge 2,\ u_0 = 1, u_1 = k. \tag{4.3}$$

If we accept this pattern, then

$$u_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-j}{j} k^{n-2j} \tag{4.4}$$

as a generalization of the corresponding well-known result for Fibonacci numbers.

Proof. The proof follows by induction on *n* with the inductive step as follows.

$$\begin{aligned} ku_{n-1}+u_{n-2} &= k\sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-j-1}{j}k^{n-2j-1}+\sum_{j=0}^{\lfloor \frac{n-2}{2}\rfloor}\binom{n-j-2}{j}k^{n-2j-2} \\ &= \sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-j-1}{j}k^{n-2j}+\sum_{j=0}^{\lfloor \frac{n-2}{2}\rfloor}\binom{n-j-2}{j}k^{n-2j-2} \\ &= \sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-j-1}{j}k^{n-2j-1}+\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-j-1}{j}k^{n-2j} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-j}{j}k^{n-2j} \\ &= u_n. \end{aligned}$$

Thus the general term of these sequences is

$$u_n=\frac{1}{\sqrt{k^2+4}}\left\{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^{n+1}-\left(\frac{k-\sqrt{k^2+4}}{2}\right)^{n+1}\right\}. \tag{4.5}$$

We note in passing that when $k=2^r$ we encounter another generalization of the Pell sequences as noted by Shannon and Horadam [15].

## 5. CONCLUDING COMMENTS

There is plenty of scope for further investigations and it is appropriate to outline two of these since this is a conference presentation. Firstly, if we consider the array,

$$R_k=\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 0 & 0 & 0 & \dots \\ 2 & 1 & 2 & 0 & 0 & \dots \\ 2 & 1 & 1 & 2 & 0 & \dots \\ 2 & 1 & 1 & 1 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{k\times k}$$

and post-multiply it by $A^k$ we find the rising diagonal sums form a generalized Lucas sequence:

$$u_n(k)\equiv u_n=ku_{n-1}+u_{n-2}-(k-1),\ \ u_0=3,\ \ u_1=2k+2, \tag{5.1}$$

with

$$u_n(1)=L_{n+2}.$$

For example,

$$R_4 A^4 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 16 & 8 & 1 & 0 \\ 64 & 48 & 12 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 10 & 2 & 0 & 0 \\ 38 & 17 & 2 & 0 \\ 150 & 105 & 25 & 2 \end{bmatrix}.$$

In addition one can use various arrays in Bondarenko [3] for higher order generalizations of the Fibonacci sequence. More immediately, readers might like to find a closed form expression for the cumulative column sums of $A^k$, that is, for $\sum_{i=0}^{n} a_{i,j}^{(k)}, n < k$, analogous to the row sums:

$$\sum_{j=1}^{i} a_{i,j}^{(k)} = (k+1)^i.$$

Secondly, consider the multiplicative analogy of (2.2) by defining:

$$v_n^{(k+1)} = v_n^{(k)} v_{n+1}^{k}, k \ge 0, n > 0, \tag{5.2}$$

with initial condition $v_n^{(0)} = F_n$. Thus

$$v_n^{(1)} = F_n F_{n+1}. \tag{5.3}$$

This arises quite naturally from the work of Ferns [7]. One can then proceed by taking logarithms of (5.2), or by letting

$$w_n^{(k+1)} = w_n^{(k)} w_{n+1}^{(k)}, k \ge 0, n > 0, \tag{5.4}$$

with initial condition $w_n^{(0)} = L_n$, and so

$$w_n^{(1)} = L_n L_{n+1}. \tag{5.5}$$

Repeated applications of (5.3) and (5.5) lead respectively to

$$v_n^{(1)} = v_{n-1}^{(1)} + v_{n-2}^{(1)} - v_{n-3}^{(1)} + F_{2n-2}, \tag{5.6}$$

and

$$w_n^{(1)} = 3v_n^{(1)} + 3v_{n-1}^{(1)} + F_{2n-1} + (-1)^n. \tag{5.7}$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $v_n^{(1)}$ | 1 | 2 | 6 | 15 | 40 | 104 | 273 |
| $w_n^{(1)}$ | 3 | 12 | 28 | 77 | 198 | 522 | 1363 |

Table 4: Multiplicative sequences

For example,

$$\begin{aligned} 3v_7^{(1)} + 3v_6^{(1)} + F_{13} - 1 &= 819 + 312 + 233 - 1 \\ &= 1363 = w_7^{(1)}. \end{aligned}$$

Again, repeated applications of (5.6) and (5.7) can be used to establish that both sequences satisfy the same fifth order homogeneous linear recurrence relation (with an attractive symmetry among the coefficients), namely,

$$v_n^{(1)} = 4v_{n-1}^{(1)} - 3v_{n-2}^{(1)} - 3v_{n-3}^{(1)} + 4v_{n-4}^{(1)} - v_{n-5}^{(1)}. \tag{5.8}$$

For instance,

$$\begin{aligned} 4v_5^{(1)} - 3v_4^{(1)} - 3v_3^{(1)} + 4v_2^{(1)} - v_1^{(1)} &= (792 + 48) - (231 + 84 + 3) \\ &= 522 = w_6^{(1)}. \end{aligned}$$

Bernoulli-Binet [17] forms can also be developed:

$$v_n^{(1)} = \tfrac{1}{5}\left(L_{2n+1} - (-1)^n\right),$$

and

$$w_n^{(1)} = L_{2n+1} + (-1)^n,$$

and more generally

$$v_n^{(k)} = F_n F_{n+k+1} \prod_{j=1}^{k} F_{n+j}^{k+1}.$$

For example,

$$\begin{aligned} v_n^{(0)} &= F_n F_{n+1}, \\ v_n^{(1)} &= F_n F_{n+1}^2 F_{n+2}, \\ v_n^{(2)} &= F_n F_{n+1}^3 F_{n+2}^3 F_{n+3}. \end{aligned}$$

These numbers can also be related to generalized binomial coefficients [8,13,19], and thence to other Fibonacci [10] and Lucas [4] identities. Moreover, if we utilize the notational convention $f_n = F_{n+1}$, then we can find that the Hadamard product of the Fibonacci generating functions for $\{f_n\}$ and $\{F_n\}$ is the generating function for $\{u_n^{(1)}\}$ [14]. This can obviously be extended indefinitely.

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## REFERENCES


1. M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions*. New York: Dover Publications, Ninth Prinitng, 1970.
2. Br U. Alfred (Brousseau). "Seeking the Lost Goldmine or Exploring for Fibonacci Factorizations." *The Fibonacci Quarterly* **3.2** (1965): 129-130.
3. B.A. Bondarenko. *Generalized Pascal Triangles and Pyramids.* (Richard A Bollinger (trans.) Santa Clara: The Fibonacci Association, 1993.
4. T. Brennan. "Problem H15." *The Fibonacci Quarterly* **1.1** (1963): 47.
5. L. Carlitz and J. Riordan. "Two Element Lattice Permutation Numbers and their *q*-Generalizations." *Duke Mathematical Journal* **31.4** (1964): 371-388.
6. M. A. Feinberg. "Lucas's Triangle." *The Fibonacci Quarterly* **5.5** (1967): 486-490.
7. H.H. Ferns. "Products of Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **7.1** (1969): 1-13.
8. H.W. Gould. "The Bracket Function and Fontené-Wrad Generalized Binomial Coefficients with Applications to Fibonomial Coefficients." *The Fibonacci Quarterly* **7.1** (1979): 23-40,55.
9. H.W. Gould & H.W. Greig. "A Lucas Triangle Primality Criterion due to that of Mann-Shanks." *The Fibonacci Quarterly* **23.1** (1985): 66-69.
10. V.E. Hoggatt Jr. "Fibonacci Numbers and Generalized Binomial Coefficients." *The Fibonacci Quarterly* **5.4** (1967): 383-400.
11. A.F. Horadam. "Generating Identities for Generalized Fibonacci and Lucas Triples." *The Fibonacci Quarterly* **15.4** (1977): 289-292.
12. H. Hosoya. "Fibonacci Triangle." *The Fibonacci Quarterly* **14.2** (1976): 173-179.
13. D. Jarden. *Recurring Sequences*. Jerusalem: Riveon Lematematika, 1966.
14. S.K. Lando. *Lectures on Generating Functions.* Providence, R.I.: American Mathematical Society, 2003.
15. A.G. Shannon and A.F. Horadam. "Generalized Pell Numbers and Polynomials." In F. Howard (ed.) *Applications of Fibonacci Numbers, Volume 9*. Dordrecht: Kluwer, pp. 213-224.
16. N.J.A. Sloane and S. Plouffe. *Encyclopedia of Integer Sequences*. New York: Academic Press, 1995.
17. G.J. Tee. "Russian Peasant Multiplication and Egyptian Division in Zeckendorf Arithmetic." *The Australian Mathematical Society Gazette* **30.5**, 2003: 267-276.
18. V. Thébault. *Les Récréations Mathématiques.* Paris: Gauthier-Villars, 1952.
19. R.F. Torretto and J.A. Fuchs. "Generalized Binomial Coefficients." *The Fibonacci Quarterly* **2.4** (1964): 296-302.


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