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AN EXTREMAL PROBLEM RELATED TO THE FIBONACCI SEQUENCE

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ABSTRACT

This paper continues our study of Fibonacci inequalities [1]. For the set $A_n = \{F_{n-1}, 4F_{n-2}, ..., (n-2)^2 F_2\}$ with k^{th} element given by $a_k = k^2 F_{n-k}$, it is proved that the unique maximal element is given by $a^* = a_4 = 16F_{n-4}$, $n \geq 9$.

1. INTRODUCTION

Here we shall discuss an extremal problem related to Fibonacci numbers, defined in terms of $\{F_m\}_{n=1}^{\infty}$ such that

$$F_{n+2} = F_{n+1} + F_n,\ F_1 = 1, F_2 = 1 \tag{1.1}$$

for each natural number *n*. For notational convenience and extensions we can allow unrestricted values of *n* [5].

The purpose of this note it to establish the maximal element of the following set. Let the natural number $n \geq 9$ be given. We construct the set

$$A_n = \{F_{n-1}, 4F_{n-2}, ..., (n-2)^2 F_2\} \tag{1.2}$$

Then the k^{th} member of the set is

$$a_k = k^2 F_{n-k}, \tag{1.3}$$

where $1 \le k \le n-2$.

2. EXAMPLES

For example,

$$A_9 = \{1\times 21, 4\times 13, 9\times 8, 16\times 5, 25\times 3, 36\times 2, 49\times 1\}, \tag{2.1}$$

and the maximal member of the set is

$$a_4 = 80;$$

similarly,

$$A_{10} = \{1\times 34, 4\times 21, 9\times 13, 16\times 8, 25\times 5, 36\times 3, 49\times 2, 64\times 1\}, \tag{2.2}$$

and the maximal element of the set is

$$a_4 = 128;$$

and

$$A_{11} = \{1\times 55, 4\times 34, 9\times 21, 16\times 13, 25\times 8, 36\times 5, 49\times 3, 64\times 2, 81\times 1\}, \tag{2.3}$$

with maximal element

$$a_4 = 208.$$

From these examples we see that the order of the elements of the set is

$$F_{n-1} < 4F_{n-2} < 9F_{n-3} < 16F_{n-4} > 25F_{n-5} > ... > (n-1)^2 F_2. \tag{2.4}$$

We now prove the result that the maximal element a^* of A_n satisfies

$$a^* = a_4 = 16F_{n-4}, n \ge 9. \tag{2.5}$$

3. MAXIMAL ELEMENT

Now let us assume the existence of a natural number q for which

$$a_q < a_{q-1},$$
$$a_q < a_{q+1}.$$

Hence,

$$q^2 F_{n-q} < (q-1)^2 F_{n-q+1},$$
$$q^2 F_{n-q} < (q+1)^2 F_{n-q-1}. \tag{3.1}$$

Both inequalities are strong because from the obvious inequality

$$2F_k > F_{k+1}, \tag{3.2}$$

it follows for the second inequality of (3.1) that

$$2q^2 F_{n-q} < 2(q+1)^2 F_{n-q-1} < (q+1)^2 F_{n-q};$$

that is,

$$2q^2 < (q+1)^2,$$

which is valid only for q=1,2,3, and hence only for these values of q is it possible for (3.1) to be valid. But for q=1, (3.1) has the form

$$F_{n-1} < 0,$$
$$F_{n-1} < 4F_{n-2},$$

which is impossible. For q=2, (3.1) has the form

$$4F_{n-2} < F_{n-1},$$
$$4F_{n-2} < 9F_{n-3},$$

which is also impossible. For q=3, (3.1) has the form

$$9F_{n-3} < 4F_{n-2},$$
$$9F_{n-3} < 16F_{n-4},$$

which again is impossible. Therefore, no natural number satisfies (3.1). Hence the set A_n has exactly one maximal element.

Let the natural number q satisfy

$$a_q > a_{q-1},$$
$$a_q > a_{q+1};$$

that is,

$$q^2 F_{n-q} > (q-1)^2 F_{n-q+1},$$
$$q^2 F_{n-q} > (q+1)^2 F_{n-q-1}. \tag{3.3}$$

For every natural number k, it follows from (3.2) and the second inequality of (3.3) that

$$2q^2 F_{n-q-1} > q^2 F_{n-q} > (q+1)^2 F_{n-q-1};$$

that is,

$$q^2 - 2q - 1 > 0,$$

which is valid for $q > 1+\sqrt{5}$; that is, $q \geq 4$.

On the other hand, if

$$2F_{k+1} > 3F_k$$

for every natural number *k*, then from the first inequality of (3.3) it follows that

$$2q^2 F_{n-q+1} 3q^2 F_{n-q} > 3(q-1)^2 F_{n-q+1};$$

that is,

$$2q^2 > 3(q-1)^2,$$

and

$$q^2 - 6q + 2 < 0,$$

which is valid for $q < 3+\sqrt{6}$; that is, $q \leq 5$. Therefore, the only possible solutions are q=4 and q=5.

Finally, for $n \geq 9$ we obtain sequentially:

$$\begin{aligned} 16F_{n-4} - 25F_{n-5} &= -9F_{n-5} + 16F_{n-6} \\ &= 7F_{n-6} - 9F_{n-7} \\ &= -2F_{n-7} + 7F_{n-8} \\ &= 5F_{n-8} - 2F_{n-9} \\ &\geq 3F_{n-9} \\ &\geq 3 \\ &> 0, \end{aligned}$$

that is, the validity of the order of the elements of A_n has been established and the maximal element of the set is $a^* = a_4 = 16F_{n-4}, n \geq 9$. For n<9, the left hand side is negative as can be verified by substitution.

To illustrate that $a_4 = \max A_n$, $n \geq 9$, is the unique maximal element, suppose $a_q > \{a_{q-1}, a_{q+1}\}$, i.e.,

$$q^2 F_{n-q} > (q+1)^2 F_{n-q-1}$$

and

$$q^2 F_{n-q} > (q-1)^2 F_{n-q+1}.$$

then

$$\frac{F_{n-q}}{F_{n-q-1}} > \left(\frac{q+1}{q}\right)^2 \tag{3.4}$$

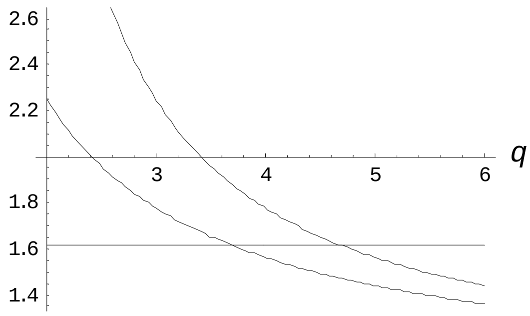
and

$$\frac{F_{n-q+1}}{F_{n-q}} < \left(\frac{q}{q-1}\right)^2 \tag{3.5}$$

must hold simultaneously. Since $\frac{F_m}{F_{m-1}} \to \phi \approx 1.618$ as $m \to \infty$, for n sufficiently large these are both achievable only if

$$\left(\frac{q+1}{q}\right)^2 < \phi < \left(\frac{q}{q-1}\right)^2 .$$

Plotting these gives:



from which, noting the monotonicity of the functions, it can be seen that the only integer solution is $q = 4$.

Further, since $\left|\frac{F_m}{F_{m-1}} - \phi\right| \to 0$ monotonically and $\frac{25}{16} < \frac{F_5}{F_4} < \frac{16}{9}$, both (3.4) and (3.5) are then satisfied for all $n-4 \geq 5$, i.e. $n \geq 9$. (That $n = 9$ is "sufficiently large" can be seen by also plotting the worst case relevant approximations to ϕ used in (3.4) and (3.5), i.e. $\frac{F_5}{F_4} = 1.\dot{6}$ and $\frac{F_6}{F_5} = 1.6$.)

4. CONCLUDING COMMENTS

There are numerous results in the literature which consider aspects of Fibonacci inequalities more generally. A sample of such types is listed in [1]. Other relevant references may be found in [2,3,4,7]. We conclude with an approximate approach to achieve the main result in another way:

For $g_k > 0$ and $n>k>0$, let

$$a_k = g_k F_{n-k}.$$

Then

$$a_k' = g_k F_{n-k}' + g_k' F_{n-k} = 0$$

when

$$\frac{g_k'}{g_k} = -\frac{F_{n-k}'}{F_{n-k}} \approx \ln \phi,$$

since, for n sufficiently large,

$$F_n \approx \frac{\phi^n}{\sqrt{5}}.$$

in which ϕ is the golden ratio. Thus the extremal position for $g_k = k^2$ is approximated by

$$k = \frac{2}{\ln \phi} \approx 4.16.$$

For the interested reader to extend the results further consider Table 1. For

$$g_k = (k+d)^m F_{n-k},$$

the analogous analysis yields

$$k = \frac{m}{\ln \phi} - d.$$

The seemingly anomalous 1s which appear for d = -3,-4,-5 appear for any even power and arise for $k+d \le 1$ because g_k is not strictly increasing. These anomalies would be removed if A_n was restricted to $k+d>1$; see the Mathematica [6] output below up to $k=n-1$.

d	$(k+d)F_{n-k}$	$n>$	$(k+d)^2 F_{n-k}$	$n>$	$(k+d)^3 F_{n-k}$	$n>$
-5	7	11	1	13	11	15
-4	6	10	1	12	10	14
-3	5	9	1	11	9	13
-2	4	8	6	10	8	12
-1	3	7	5	9	7	11
0	2	6	4	8	6	10
1	1	5	3	7	5	9
2	1	4	2	6	4	8
3	1	3	1	5	3	7
4	1	2	1	4	2	6

Table 1: Maximum element position

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APPENDIX

The tables below show A_n, maximum value and {position of maximum value} for $g_k=(k+d)^2$

$d = 0$

{1}	1	{1}
{1, 4}	4	{2}
{2, 4, 9}	9	{3}
{3, 8, 9, 16}	16	{4}
{5, 12, 18, 16, 25}	25	{5}
{8, 20, 27, 32, 25, 36}	36	{6}
{13, 32, 45, 48, 50, 36, 49}	50	{5}
{21, 52, 72, 80, 75, 72, 49, 64}	80	{4}
{34, 84, 117, 128, 125, 108, 98, 64, 81}	128	{4}
{55, 136, 189, 208, 200, 180, 147, 128, 81, 100}	208	{4}
{89, 220, 306, 336, 325, 288, 245, 192, 162, 100, 121}	336	{4}
{144, 356, 495, 544, 525, 468, 392, 320, 243, 200, 121, 144}	544	{4}
{233, 576, 801, 880, 850, 756, 637, 512, 405, 300, 242, 144, 169}	880	{4}
{377, 932, 1296, 1424, 1375, 1224, 1029, 832, 648, 500, 363, 288, 169, 196}	1424	{4}

Note that in the following tables, extremal values occur four places to the right of the 0 for *n* sufficiently large.

$d = -1$

{0}	0	{1}
{0, 1}	1	{2}
{0, 1, 4}	4	{3}
{0, 2, 4, 9}	9	{4}
{0, 3, 8, 9, 16}	16	{5}
{0, 5, 12, 18, 16, 25}	25	{6}
{0, 8, 20, 27, 32, 25, 36}	36	{7}
{0, 13, 32, 45, 48, 50, 36, 49}	50	{6}
{0, 21, 52, 72, 80, 75, 72, 49, 64}	80	{5}
{0, 34, 84, 117, 128, 125, 108, 98, 64, 81}	128	{5}
{0, 55, 136, 189, 208, 200, 180, 147, 128, 81, 100}	208	{5}
{0, 89, 220, 306, 336, 325, 288, 245, 192, 162, 100, 121}	336	{5}
{0, 144, 356, 495, 544, 525, 468, 392, 320, 243, 200, 121, 144}	544	{5}
{0, 233, 576, 801, 880, 850, 756, 637, 512, 405, 300, 242, 144, 169}	880	{5}

$d = -2$

{1}	1	{1}
{1, 0}	1	{1}
{2, 0, 1}	2	{1}
{3, 0, 1, 4}	4	{4}
{5, 0, 2, 4, 9}	9	{5}
{8, 0, 3, 8, 9, 16}	16	{6}
{13, 0, 5, 12, 18, 16, 25}	25	{7}
{21, 0, 8, 20, 27, 32, 25, 36}	36	{8}
{34, 0, 13, 32, 45, 48, 50, 36, 49}	50	{7}
{55, 0, 21, 52, 72, 80, 75, 72, 49, 64}	80	{6}
{89, 0, 34, 84, 117, 128, 125, 108, 98, 64, 81}	128	{6}
{144, 0, 55, 136, 189, 208, 200, 180, 147, 128, 81, 100}	208	{6}
{233, 0, 89, 220, 306, 336, 325, 288, 245, 192, 162, 100, 121}	336	{6}
{377, 0, 144, 356, 495, 544, 525, 468, 392, 320, 243, 200, 121, 144}	544	{6}

$d = -3$

{4}	4	{1}
{4, 1}	4	{1}
{8, 1, 0}	8	{1}
{12, 2, 0, 1}	12	{1}
{20, 3, 0, 1, 4}	20	{1}
{32, 5, 0, 2, 4, 9}	32	{1}
{52, 8, 0, 3, 8, 9, 16}	52	{1}
{84, 13, 0, 5, 12, 18, 16, 25}	84	{1}
{136, 21, 0, 8, 20, 27, 32, 25, 36}	136	{1}
{220, 34, 0, 13, 32, 45, 48, 50, 36, 49}	220	{1}
{356, 55, 0, 21, 52, 72, 80, 75, 72, 49, 64}	356	{1}
{576, 89, 0, 34, 84, 117, 128, 125, 108, 98, 64, 81}	576	{1}
{932, 144, 0, 55, 136, 189, 208, 200, 180, 147, 128, 81, 100}	932	{1}
{1508, 233, 0, 89, 220, 306, 336, 325, 288, 245, 192, 162, 100, 121}	1508	{1}