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On Some Solved and Unsolved Problems Related to Egyptian Unit Fractions

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Abstract. In the paper a method named "Continuation of Unity" that is very useful for obtaining infinitely many decompositions of the number 1 with the help of Unit Egyptian fractions starting from given initial decomposition is demonstarted.The important question about decompositions of the number 1 with the help of unit fractions with distinct denominators that belong to a fixed increasing arithmetic progression of natural numbers is discussed. Also some applications related to fractions with odd denominators only are proposed.

Key Words: Egyptian fraction, Unit fraction, Abundant number, Pseudoperfect number.

Let $n \geq 2, x_i \in Z^+, i = \overline{1,n}$ and $x_i < x_{i+1}, i = \overline{1,n-1}$, where here and further Z^+ denotes the set of all positive integers. In this case we shall call the n-tuple $X_n\colon =< x_1, ..., x_n >$ a decomposition of the number 1 and we shall write $X_n \in D(1)$ if and only if the equality:

$$1 = \sum_{i=1}^{n} \frac{1}{x_i} \qquad (1),$$

holds. Also, we say that X_n has a length n.

Below we propose a simple idea that gives us a possibility to generate infinitely many decompositions with increasing lengths beginning from a single one.

Namely, let $X_n \in D(1)$. Then we rewrite (1) in the form:

$$1 = \sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n}.1$$

After that we substitute the factor 1 multiplied by $\frac{1}{x_n}$ with $\sum_{i=1}^{n} \frac{1}{x_i}$ due to (1) and obtain:

$$1 = \sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n}.(\sum_{i=1}^{n} \frac{1}{x_i}) \qquad (2)$$

From (2) it is seen that the equality

$$1 = \sum_{i=1}^{n-1} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i.x_n}$$

holds. Hence $Y_{2n-1} \in D(1)$, where $Y_{2n-1} =< y_1, ..., y_{2n-1} >$ has a length $2n-1$ and is expressed through the components of X_n using the equalities:

$$y_i = x_i, \text{ for } i = \overline{1, n-1}; y_{n-1+i} = x_i.x_n \text{ for } i = \overline{1, n}.$$

In the same manner from Y_{2n-1} we can obtain another decomposition of the number 1, i.e. $Z_{3n-2} \in D(1)$ with length $3n-2$. As it can be easily seen the process can be continued indefinitely.

We shall call the method described above, for brevity, the method of continuation of unity or in short **CU**.

Further we want to consider the decompositions of number 1, i.e. $Y_n =< y_1, ..., y_n >\in D(1)$ such that the components $y_i, i = \overline{1, n}$ are some terms of an infinite arithmetic progression given by:

$$\{am + b\}_{m=1}^{m=\infty},$$

where $a \geq 2 \in Z^+$ and $b \in \overline{1, a-1}$ are fixed numbers. We will denote the infinite (accountable) set whose elements are all terms of this progression by: $A(a, b)$.

Below we shall give some results.

Remark1. Obviously, a necessary condition for $Y_n \in D(1)$, when $y_i \in A(a, b), i = \overline{1, n}$, is

$$\textbf{gcd } (a, b) = 1 \qquad (3),$$

where here and further **gcd** $(a, b) = 1$ denotes the greatest common divisor of the numbers a and b. **Theorem 1.** The necessary condition for the existence of $Y_n \in D(1)$, with $y_i \in A(a, b), i = \overline{1, n}$ is the simultaneous validity of (3) and of the congruence:

$$n \equiv b (\text{mod } a) \qquad (4).$$

Proof. Let $Y_n \in D(1)$ and $y_i \in A(a, b), i = \overline{1, n}$. Then (3) is true (see Remark 1.) and we have $y_i = a.x_i + b$, for an appropriate $x_i \in Z^+$. Also the equality:

$$1 = \sum_{i=1}^{n} \frac{1}{a.x_i + b},$$

holds. The last equality implies that:

$$\prod_{i=1}^{n} (a.x_i + b) = \sum_{i=1}^{n} \frac{\prod_{j=1}^{n} (a.x_j + b)}{a.x_i + b}.$$

Hence

$$b^n \equiv n.b^{n-1} (\text{mod } a).$$

The last congruence provides (4) because of (3).

The Theorem 1 is proved.

We will require also the following:

Theorem 2. Let $a \geq 2, n \in Z^+$ be fixed, and $n \equiv 1$ (mod a). If there exists at least one $X_n \in D(1)$ with $x_i \in A(a, 1), i = \overline{1, n}$, then there exists an infinite sequence $\{X_{k.n-k+1}\}_{k=1}^{k=\infty}$ of decompositions of number 1 such that the k- th term of this sequence satisfies the conditions: $X_{k.n-k+1} \in D(1), x_i \in A(a, 1), i = \overline{1, n-k+1}$, and $X_k n - k + 1$ has length $k.n - k + 1$.

Proof. Since we are in the case $b = 1$ and the congruence from the condition of the Theorem 2 is true for the given $n \in Z^+$, let us assume that for the same n there exists $X_n \in D(1)$ and $x_i \in A(a, 1), i = \overline{1, n}$. Now we are able to apply the method **CU**, mentioned above, to receive sequentially $X_{2n-1}, X_{3n-2}, ..., X_{k.n-k+1}$, etc. $\in D(1)$. The components of all these decompositions of the number 1 are elements of $A(a, 1)$.That is seen from the description of the applied method **CU**.

The Theorem 2 is proved.

Theorem 3. Let $a, n, s \in Z^+, a \geq 2, n \geq 2, s \geq 2$ be fixed and the congruences

$$s \equiv 1 \text{ (mod } a);$$

$$n \equiv b \pmod{a}$$

hold, and (3) be fulfilled, where $b \in i = \overline{1,a}$ is fixed . Then, if there exist at least one $X_n \in D(1)$, with $x_i \in A(a,b), i = \overline{1,n}$ and at least one $Y_s \in D(1)$ with $y_i \in A(a,1), i = \overline{1,s}$ we can find infinitely many decompositions of the number 1 with increasing lengths whose components belong to $A(a,b)$.

Proof. We will show how with the aid of the method **CU** on the first step one can find one of the mentioned infinitely many decompositions of the number 1 with length $n+s-1$ and with the requested by the Theorem 3 form. The rest could be obtained in the same manner using the method **CU** sequentially. Since (3) and the congruence $s \equiv 1 \pmod{a}$ are necessary conditions for the existence of $X_n \in D(1)$ with $x_i \in A(a,b), i = \overline{1,n}$ and $Y_s \in D(1)$ with $y_i \in A(a,1), i = \overline{1,s}$, and these necessary conditions are provided from the condition of Theorem 3, let us assume that there exist such decompositions of the number $1, X_n$ and Y_s. Hence,

$$1 = \sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n}.1$$

and

$$1 = \sum_{i=1}^{s} \frac{1}{y_i}.$$

Using the method **CU** from the above two equalities we obtain:

$$1 = \sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{x_n} \sum_{j=1}^{s} \frac{1}{y_j} = \sum_{i=1}^{n+s-1} \frac{1}{z_i}$$

where $z_i = x_i$ for $i = \overline{1,n-1}$ and $z_{n-1+i} = y_i.x_n, i = \overline{1,s}$. Hence, $Z_{n+s-1} =< z_1,..,z_{n+s-1} >\in D(1)$. But it can be checked directly that $z_i \in A(a,b)$ and thus the desired decomposition that is based on X_n and Y_s is constructed. By the noted above, Theorem 3 is proved.

Summary of the above results.

For fixed $a \in Z^+, a \geq 2$, gcd $(a,b) = 1$ and $b \in \overline{1,a-1}$ and for fixed $n,s \in Z^+, n \geq 2, s \geq 2$ let the congruences:

$$n \equiv b \pmod{a}; s \equiv 1 \pmod{a},$$

hold. If there exists $Y_s \in D(1)$ with $y_i \in A(a,1), i = \overline{1,s}$, then there exist infinitely many decompositions of number 1 with increasing lengths whose components belong to $A(a,1)$, and if there exists $X_n \in D(1)$ with $x_i \in A(a,b), i = \overline{1,n}$ then there exist infinitely many decompositions of the number 1 with increasing lengths such that all their components belong to $A(a,b)$.

Naturally we must put the following two open problems:

Open Problem 1. For a given $a \in Z^+, a \geq 2$, does there always exist $Y_s \in D(1)$ with $y_i \in A(a,1)$ and s satisfies the congruence $s \equiv 1 \pmod{a}$?

Open Problem 2. For a given $a, b \in Z^+, a \geq 2, b \in \overline{1, a-1}$, gcd $(a,b) = 1$ does there always exist $X_n \in D(1)$ with $x_i \in A(a,b)$ and n satisfies the congruence $n \equiv b \pmod{a}$?

Further as a particular case of the above considerations we study the special case when $a = 2, b = 1$. Then $A(2,1)$ coincides with the set of all odd numbers. For this reason in this case we look for decompositions of the number 1 using different unit fractions with odd denominators only. From (3) it is seen that n must be odd too. In [1] is proved that if $X_n \in D(1)$ with $x_i \in A(2,1)$ (i.e. x_i are distinct odd numbers)$i = \overline{1,n}$ then $n \geq 9$ and n is an odd number. For $n = 9$ we have only five different decompositions of number one. They are the following:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{231}$$
$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{21} + \frac{1}{231} + \frac{1}{315}$$
$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{33} + \frac{1}{45} + \frac{1}{385}$$
$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{21} + \frac{1}{165} + \frac{1}{693}$$
$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{21} + \frac{1}{135} + \frac{1}{10395}$$

In all these decompositions the repeating numbers that take part as denominators are $3, 5, 7, 9, 11, 15$. The second of the above decompositions corresponds to the number 3465. As it is shown in [1] this is the smallest odd number $k > 1$ that can be represented as a sum of the fewest in number among all of its proper divisors. Namely 3465 is a sum of nine in number among all its proper divisors:

$$3465 = 11 + 15 + 165 + 231 + 315 + 385 + 495 + 1155$$

Below we answer positively to the Open problem 1 and Open problem 2 for the special case: $a = 2, b = 1$.

Theorem 4. For every odd $n \geq 9$ there exists $X_n \in D(1)$ with x_i distinct odd numbers,

$i = \overline{1, n}$.

Proof. As it was mentioned above, for $n = 9$ we have exactly five in number different decompositions of the number 1 with different odd denominators only. Now we use the trivially checked identity:

$$\frac{1}{3k} = \frac{1}{5k} + \frac{1}{9k} + \frac{1}{45k} \tag{5}$$

that is valid for every $k \in Z^+$. Let us consider the equality:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{21} + \frac{1}{231} + \frac{1}{315} \qquad (6),$$

Since $315 = 3k$ for $k = 105$ we may replace $\frac{1}{315}$ from the right hand-side of (6) with the help of (5). Thus we obtain a new decomposition of the number 1 using eleven in number different unit fractions with odd denominators. The last denominator would also be of the kind $3k$ which allows us to use (5) again and to receive a new decomposition of the number 1 using thirteen in number different unit fractions with odd denominators. The process may be continued as far as we want since the last denominator of the received decomposition will always be divisible by 3. In this manner we obtain infinitely many decompositions of the number 1 using unit fractions with different odd denominators whose lengths cover all odd numbers $n \geq 9$. The Theorem 4 is proved.

Let us consider the decomposition:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{45} + \frac{1}{135} + \frac{1}{315}$$

This decomposition corresponds to the representation:

$$945 = 3 + 7 + 21 + 27 + 35 + 45 + 63 + 105 + 135 + 189 + 315 \tag{7}$$

In the right hand-side of (7) exactly eleven in number of all proper divisors of the number $945 = 3^3.5.7$ are taking part. Thus 945 is odd pseudoperfect number (pseudoperfect numbers are also called semiperfect numbers). Moreover, this is the smallest odd abundant number and pseudoperfect number too. For this fact one may see [2]. That was established for the first time by Abu Mansur ibn Tahir Al-Baghdadi (980-1037). In Europe around 1600 the same fact was noted by Bachet de Méziriac see [3]. The first three odd pseudoperfect and simultaneously abundant numbers are $945, 1575, 2205$..

Remark 2. There is still unproved conjecture of P. Erdös that the set of all odd pseudoperfect numbers coincides with the set of all odd abundant numbers.

Another approach (that we propose here) for finding decompositions of number 1 using unit fractions with odd different denominators whose lengths form an increasing sequence is the following:

For a given odd $n \geq 9$ and decomposition (1) we start from the easily verified identity:

$$\frac{1}{x_n} = \frac{1}{2x_n+1} + \frac{1}{x_n.(2x_n+1)} + \sum_{i=1}^{x_n} \frac{1}{x_n.(2i-1)(2i+1)}. \tag{8}$$

Substituting (8) into (1), we obtain a new decomposition of the number 1, with greater length than that in (1), which uses unit fractions with odd denominators only. It is easy to see that these denominators are different if and only if the number $\frac{n+1}{2}$ is not a square. The process may be continued as far as we want and as a result are obtained the decompositions of the number 1 using different unit fractions with odd denominators whose lengths form an increasing sequence. For the interested reader we also recommend [4] thematically related to the contents of the present paper.

References

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