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ON TWO NEW 2-FIBONACCI SEQUENCES

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**To Prof. Ivan Dotsinsky
for his 70-th birthday!**

Let the arbitrary real numbers $a, b, c,$ and d be given.

In [1, 2] four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described. We shall call them 2-Fibonacci sequences (or 2-F-sequences). The four schemes are the following

$$\begin{aligned} &\alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d \\ &\alpha_{n+2} = \beta_{n+1} + \beta_n, \ n \geq 0 \\ &\beta_{n+2} = \alpha_{n+1} + \alpha_n, \ n \geq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &\alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d \\ &\alpha_{n+2} = \alpha_{n+1} + \beta_n, \ n \geq 0 \\ &\beta_{n+2} = \beta_{n+1} + \alpha_n, \ n \geq 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &\alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d \\ &\alpha_{n+2} = \beta_{n+1} + \alpha_n, \ n \geq 0 \\ &\beta_{n+2} = \alpha_{n+1} + \beta_n, \ n \geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &\alpha_0 = a, \ \beta_0 = b, \ \alpha_1 = c, \ \beta_1 = d \\ &\alpha_{n+2} = \alpha_{n+1} + \alpha_n, \ n \geq 0 \\ &\beta_{n+2} = \beta_{n+1} + \beta_n, \ n \geq 0 \end{aligned} \tag{4}$$

In [1] L. Atanassova, D. Sasselov, and the author discussed scheme (1), while scheme (2) has been studied by the author in [2] (see, also [3-9]).

Graphically, the (n+2)-nd members of the different schemes are obtained from the n-th and the (n+1)-st members as shown in Fig. 1-4.

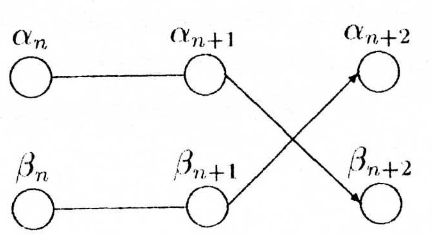

Fig. 1.

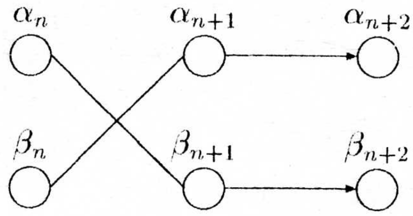

Fig. 2.

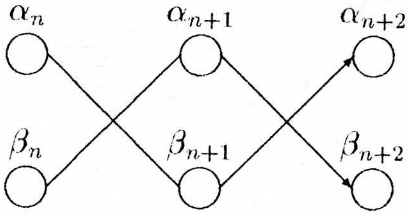

Fig. 3.

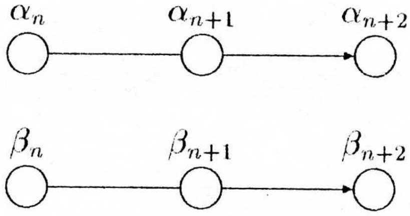

Fig. 4.

Clearly, if we set $a = b$ and $c = d$, then sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is called a generalized Fibonacci sequence, where

$$F_0(a,c) = a,\ \ F_1(a,c) = c,\ \ F_{n+2}(a,c) = F_{n+1}(a,c) + F_n(a,c).$$

Let $F_i = F_i(0,1)$; $\{F_i\}_{i=0}^{\infty}$ be the ordinary Fibonacci sequence.

Now, we shall introduce two new (mixed) schemes, which have the following recurrence forms:

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ &\alpha_{2n+2} = \beta_{2n+1} + \beta_{2n} \\ &\beta_{2n+2} = \alpha_{2n+1} + \alpha_{2n} \\ &\alpha_{2n+3} = \alpha_{2n+2} + \beta_{2n+1} \\ &\beta_{2n+3} = \beta_{2n+2} + \alpha_{2n+1} \\ &(n \geq 0) \end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\gamma_0 = a, \ \delta_0 = b, \ \gamma_1 = c, \ \delta_1 = d \\
&\gamma_{2n+2} = \gamma_{2n+1} + \delta_{2n} \\
&\delta_{2n+2} = \delta_{2n+1} + \gamma_{2n} \\
&\gamma_{2n+3} = \delta_{2n+2} + \delta_{2n+1} \\
&\delta_{2n+3} = \gamma_{2n+2} + \gamma_{2n+1} \\
&(n \geq 0)
\end{aligned} , \qquad (6)$$

and the graphical interpretations from Fig. 5 and 6.

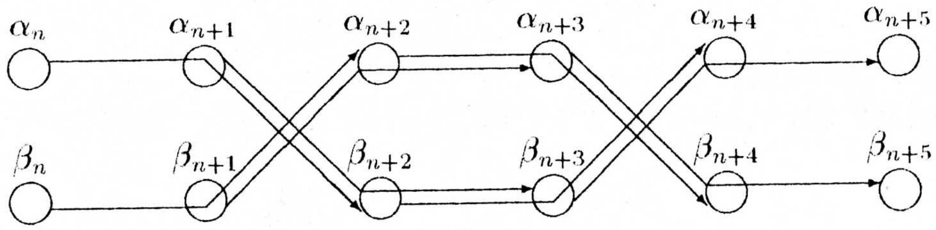

Fig. 5.

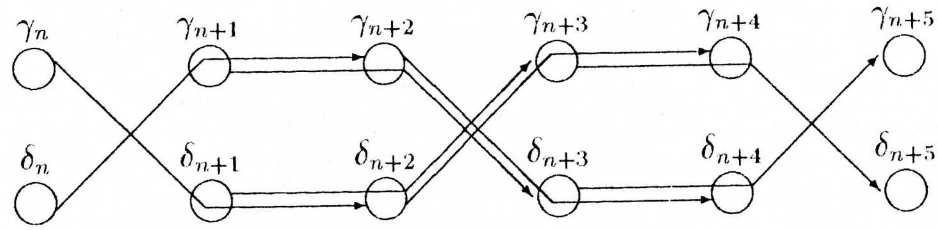

Fig. 6.

The first ten terms of the sequences defined in (5) are:

n	α_n	β_n
0	a	b
1	c	d
2	$b + d$	$a + c$
3	$b + 2d$	$a + 2c$
4	$2a + 3c$	$2b + 3d$
5	$3a + 5c$	$3b + 5d$
6	$5b + 8d$	$5a + 8c$
7	$8b + 13d$	$8a + 13c$
8	$13a + 21c$	$13b + 21d$
9	$21a + 34c$	$21b + 34d$

while these, for the sequences defined in (6) are:

n	γ_n	δ_n
0	a	b
1	c	d
2	$b+c$	$a+d$
3	$a+2d$	$b+2c$
4	$2a+3d$	$2b+3c$
5	$3b+5c$	$3a+5d$
6	$5b+8c$	$5a+8d$
7	$8a+13d$	$8b+13c$
8	$13a+21d$	$13b+21c$
9	$21b+34c$	$21a+34d$

For the members of sequences (5) and (6) are valid

THEOREM 1: If $k \geq 0$, then

$$\begin{aligned}
\alpha_{4k+2} &= F_{4k+1}b + F_{4k+2}d \\
\alpha_{4k+3} &= F_{4k+2}b + F_{4k+3}d \\
\alpha_{4k+4} &= F_{4k+3}a + F_{4k+4}c \\
\alpha_{4k+5} &= F_{4k+4}a + F_{4k+5}c \\
\beta_{4k+2} &= F_{4k+1}a + F_{4k+2}c \\
\beta_{4k+3} &= F_{4k+2}a + F_{4k+3}c \\
\beta_{4k+4} &= F_{4k+3}b + F_{4k+4}d \\
\beta_{4k+5} &= F_{4k+4}b + F_{4k+5}d
\end{aligned}$$

THEOREM 2: If $k \geq 0$, then

$$\begin{aligned}
\gamma_{4k+2} &= F_{4k+1}b + F_{4k+2}c \\
\gamma_{4k+3} &= F_{4k+2}a + F_{4k+3}d \\
\gamma_{4k+4} &= F_{4k+3}a + F_{4k+4}d \\
\gamma_{4k+5} &= F_{4k+4}b + F_{4k+5}c \\
\delta_{4k+2} &= F_{4k+1}a + F_{4k+2}d \\
\delta_{4k+3} &= F_{4k+2}b + F_{4k+3}c \\
\delta_{4k+4} &= F_{4k+3}b + F_{4k+4}c \\
\delta_{4k+5} &= F_{4k+4}a + F_{4k+5}d
\end{aligned}$$

THEOREM 3: If $n \geq 0$, then

$$\alpha_n + \beta_n = \gamma_n + \delta_n.$$

!!!!! We shall omit the proofs because they are not difficult.

References

[1] Atanassov K., L. Atanassova, D. Sasselov, A new perspective to the generalization of the Fibonacci sequence, *The Fibonacci Quarterly*, Vol. 23 (1985), No. 1, 21-28.

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[4] Spickerman W., R. Joyner, R. Creech, On the $(2, F)$-generalizations of the Fibonacci sequence, *The Fibonacci Quartarly*, Vol. 30 (1992), No. 4, 310-314.

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[8] Shannon A., R. Melham, Carlitz generalizations of Lucas and Lehmer sequences, *The Fibonacci Quartarly*, Vol. 31 (1993), No. 2, 105-111.

[9] Atanassov K., V. Atanassova, A. Shannon, J. Turner, New Visual Perspectives on Fibonacci Numbers. World Scientific, New Jersey, 2002.