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DIGIT SUM BASES FOR FIBONACCI AND RELATED NUMBERS

Krassimir T. Atanassov[1] and Anthony G. Shannon[2]

[1] CLBME - Bulgarian Academy of Sciences, P.O.Box 12, Sofia-1113, Bulgaria
e-mail: *krat@bas.bg*

[2] KvB Institute of Technology, North Sydney, 2060, & Warrane College, The University of New South Wales, Kensington, 1465, Australia
e-mail: *tony@kvb.edu.au*

In [1-4] a digital arithmetical function is defined and its properties are described. Here we shall discuss its application to Fibonacci types of sequences.

We shall use the natural number n in the following form

$$n = \sum_{i=1}^{k} a_i.10^{k-i} \equiv \overline{a_1 a_2 \ldots a_k},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq k)$.

In [1-4] a function denoted by φ (not in the sense of Euler's totient funcion) was defined by:

$$\varphi(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{i=1}^{m} a_i, & \text{if } n > 0 \end{cases}$$

Obviously, since for $k > 1$

$$\varphi(n) = \sum_{i=1}^{k} a_i < \sum_{i=1}^{k} a_i.10^{k-i} = n.$$

The following assertion is valid for it.

LEMMA: Function φ satisfies the scheme:

$$\varphi(n+1) = \begin{cases} \varphi(n)+1 & , \text{ if } a_k \neq 9 \\ \varphi(n) - 9r + 1 & , \text{ if } a_k = a_{k-1} = \ldots = a_{k-r+1} = 9 \\ & \text{ where } 1 \leq r \leq k \text{ and } a_{k-r} \neq 9 \\ 1 & , \text{ if } a_k = a_{k-1} = \ldots = a_1 = 9 \end{cases}$$

In [1-4] a sequence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots$, is defined, in which l is a natural number and where

$$\varphi_0(n) = n,$$

$$\varphi_{l+1}(n) = \varphi(\varphi_l(n)).$$

Obviously, for every $l \in \mathcal{N}$ (the set of all non-negative numbers) $\varphi_l : \mathcal{N} \to \mathcal{N}$. Then for every $n \in \mathcal{N}$, there will exist $l \in \mathcal{N}$ so that

$$\varphi_l(n) = \varphi_{l+1}(n) \in \Delta \equiv \{0, 1, 2, \ldots, 9\}.$$

Let the function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

where

$$\varphi_{l+1}(n) = \varphi_l(n).$$

Hence, $\psi : \mathcal{N} \to \mathcal{N}$.

The following assertions are easy to prove (see [1-4]).

Proposition 1: Function ψ satisfies the scheme:

$$\psi(0) = 0$$

$$\psi(n+1) = \psi(\psi(n)+1).$$

Proposition 2: For every two natural numbers m and n:

(a) $\psi(m+n) = \psi(\psi(m) + \psi(n))$,

(b) $\psi(m.n) = \psi(\psi(m).\psi(n)) = \psi(m.\psi(n)) = \psi(\psi(m).n)$,

(c) $\psi(m^n) = \psi(\psi(m)^n)$,

(d) $\psi(n+9) = \psi(n)$,

(e) $\psi(9n) = 9$.

Proposition 3: For every natural number n:

$$\psi(n) \equiv \varphi(n) \equiv n(\text{mod } 9).$$

If $\text{Mod}_9(n) \equiv n(\text{mod } 9)$ and $\text{Mod}_9 : \mathcal{N} \to \{0, 1, ..., 8\}$, then from Theorem 3 it can be directly seen that there is a remarkable resemblance between the functions ψ and Mod_9. However, the difference is also essential: $\psi : \mathcal{N} \to \{0, 1, ..., 9\}$, and, for example,

$$\psi(9) = 9 \neq 0 = \text{Mod}_9(9).$$

Let the sequence of natural numbers $a_1, a_2, ...$ be given, and let

$$c_i = \psi(a_i) \ (i = 1, 2, ...).$$

We need deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and l exist so that $l \geq 0$,

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we shall say that

$$[c_{l+1}, c_{l+2}, ..., c_{l+k}]$$

is the *base* of the sequence $c_1, c_2, ...$ with length k and with respect to the function ψ.

For example, the Fibonacci sequence $\{F_i\}_{i=0}^{\infty}$, for which

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \ (n \geq 0)$$

has a base of length 24 with respect to the function ψ, and it is the following (see [1]):

$$[1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9].$$

The Lucas sequence $\{L_i\}_{i=0}^{\infty}$, for which

$$L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \ (n \geq 0)$$

also has a base of length 24 with respect to the function ψ, and it is the following:

$$[2, 1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7, 8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8].$$

Even the Lucas-Lehmer sequence $\{l_i\}_{i=0}^{\infty}$, for which

$$l_1 = 4, l_{n+1} = l_n^2 - 2 \ (n \geq 0)$$

has a base with one element, namely [5] with respect to the function ψ.

We can construct the following table, too, for the natural number $k \geq 0$, for powers of Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$\psi(F_n^{6k+2})$	1	1	4	9	7	1	7	9	4	1	1	9	1	1	4	9	7	1	7	9	4	1	1	9
$\psi(F_n^{6k+3})$	1	1	8	9	8	8	1	9	1	1	8	9	8	8	1	9	1	1	8	9	8	8	1	9
$\psi(F_n^{6k+4})$	1	1	7	9	4	1	4	9	7	1	1	9	1	1	7	9	4	1	4	9	7	1	1	9
$\psi(F_n^{6k+5})$	1	1	5	9	2	8	7	9	4	1	8	9	8	8	4	9	7	1	2	9	5	8	1	9
$\psi(F_n^{6k+6})$	1	1	1	9	1	1	1	9	1	1	1	9	1	1	1	9	1	1	1	9	1	1	1	9
$\psi(F_n^{6k+7})$	1	1	2	9	5	8	4	9	7	1	8	9	8	8	7	9	4	1	5	9	2	8	1	9

Here we shall formulate new statements that can be proved by induction.

Theorem 1: The Pell sequence $\{P_i\}_{i=0}^{\infty}$, for which

$$P_1 = P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n\ (n \geq 0)$$

has a base of length 24 with respect to the function ψ, and it is the following:

$$[1, 2, 5, 3, 2, 7, 7, 3, 4, 2, 8, 9, 8, 7, 4, 6, 7, 2, 2, 6, 5, 7, 1, 9].$$

Theorem 2: The Tribonacci sequence $\{T_i\}_{i=0}^{\infty}$, for which

a) $T_1 = T_2 = 0, T_3 = 1$, or $T_1 = 0, T_2 = T_3 = 1$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ $(n \geq 0)$ has a base of length 39 with respect to the function ψ and it is the following:

$$[1, 1, 2, 4, 7, 4, 6, 8, 9, 5, 4, 9, 9, 4, 4, 8, 7, 1, 7, 6, 5, 9, 2, 7, 9, 9, 7, 7, 5, 1,$$
$$4, 1, 6, 2, 9, 8, 1, 9, 9].$$

b) $T_1 = T_2 = T_3 = 1, T_{n+3} = T_{n+2} + T_{n+1} + T_n$ $(n \geq 0)$ has a base of length 39 $(= 39 \times 1)$ with respect to the function ψ, and it is the following:

$$[1, 1, 1, 3, 5, 9, 8, 4, 3, 6, 4, 4, 5, 4, 4, 4, 3, 2, 9, 5, 7, 3, 6, 7, 7, 2, 7, 7, 7, 3,$$
$$8, 9, 2, 1, 3, 6, 1, 1, 8].$$

Theorem 3: The Padovan sequence $\{P_i\}_{i=0}^{\infty}$, for which

$$P_1 = P_2 = P_3 = 2, P_{n+3} = P_{n+1} + P_n \ (n \geq 0)$$

has a base of length 39 with respect to the function ψ, and it is the following:

$$[1,1,1,2,2,3,4,5,7,9,3,1,1,4,2,5,6,7,2,4,9,6,4,6,1,1,7,2,8,9,$$

$$1,8,1,9,9,1,9,1,9].$$

Theorem 4: The Tetrabonacci sequence $\{U_i\}_{i=0}^{\infty}$, for which

a) $U_1 = U_2 = U_3 = 0, U_4 = 1$, or $U_1 = U_2 = 0, U_3 = U_4 = 1$, and $U_{n+4} = U_{n+3} + U_{n+2} + U_{n+1} + U_n$ $(n \geq 0)$ has a base of length 78 with respect to the function ψ, and it is the following:

$$[1,1,2,4,8,6,2,2,9,1,5,8,5,1,1,6,4,3,5,9,3,2,1,6,3,3,$$

$$4,7,8,4,5,6,5,2,9,4,2,8,5,1,7,3,7,9,8,9,6,5,1,3,6,6,$$

$$7,4,5,4,2,6,8,2,9,7,8,8,5,1,4,9,1,6,2,9,9,8,1,9,9,9]$$

b) $U_1 = 0, U_2 = U_3 = U_4 = 1$, $U_{n+4} = U_{n+3} + U_{n+2} + U_{n+1} + U_n$ $(n \geq 0)$ has a base of length 78 with respect to the function ψ, and it is the following:

$$[1,1,1,3,6,2,3,5,7,8,5,7,9,2,5,5,3,6,1,6,7,2,7,4,2,6,$$

$$1,4,4,6,6,2,9,5,4,2,2,4,3,2,2,2,9,6,1,9,7,5,4,7,5,3,$$

$$1,7,7,9,6,2,6,5,1,5,8,1,6,2,8,8,6,6,1,3,7,8,1,1,8,9]$$

c) $U_1 = U_2 = U_3 = U_4 = 1$, $U_{n+4} = U_{n+3} + U_{n+2} + U_{n+1} + U_n$ $(n \geq 0)$ has a base of length 78 $(= 39 \times 2)$ with respect to the function ψ, and it is the following:

$$[1,1,1,1,4,7,4,7,4,4,1,7,7,1,7,4,1,4,7,7,1,1,7,7,7,4,$$

$$7,7,7,7,1,4,1,4,1,1,7,4,4,7,4,1,7,1,4,4,7,7,4,4,4,1,$$

$$4,4,4,4,7,1,7,1,7,7,4,1,1,4,1,7,4,7,1,1,4,4,1,1,1,7]$$

Theorem 5: The Pentabonacci sequence $\{V_i\}_{i=0}^{\infty}$, for which a) $V_1 = V_2 = V_3 = V_4 = 0, V_5 = 1$, or $V_1 = V_2 = V_3 = 0, V_4 = V_5 = 1$, and $V_{n+5} = V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n$ $(n \geq 0)$ has a base of length 312 $(= 39 \times 8)$ with respect to the function ψ, and it is the following:

$$[1,1,2,4,8,7,4,7,3,2,5,3,2,6,9,7,9,6,1,5,1,4,8,1,1,6,$$

$$2,9,1,1,1,5,8,7,4,7,4,3,7,7,1,4,4,5,3,8,6,8,3,1,8,8,$$

$$1,3,3,5,2,5,9,6,9,4,6,7,5,4,8,3,9,2,8,3,7,2,4,6,4,5,$$

$$3,4,4,2,9,4,5,6,8,5,1,7,9,3,7,9,8,9,9,6,5,1,3,6,3,9,$$

$$4,7,2,7,2,4,4,1,9,2,2,9,5,9,9,7,3,6,7,5,1,4,5,4,1,6,$$

$$2,9,4,4,7,8,5,1,7,1,4,9,4,7,7,4,4,8,3,8,9,5,6,4,5,2,$$

$$4,3,9,5,5,8,3,3,6,7,9,1,8,4,2,6,3,5,2,9,7,8,4,3,4,8,$$

$$9,1,7,2,9,1,2,3,8,5,1,1,9,6,4,3,5,9,9,3,2,1,6,3,6,9,$$

$$7,4,2,1,5,1,4,4,6,2,8,6,8,3,9,7,6,6,4,5,1,4,2,7,1,6,$$

$$2,9,7,7,4,2,2,4,1,4,4,6,1,7,4,4,4,2,3,8,3,2,9,7,2,5,$$

$$7,3,6,5,8,2,6,9,3,1,3,4,2,4,5,9,6,8,5,6,7,5,4,9,4,2,$$

$$6,7,1,2,9,7,8,9,8,5,1,4,9,9,1,6,2,9,9,9,8,1,9,9,9,9]$$

b) $V_1 = V_2 = 0, V_3 = V_4 = V_5 = 1, V_{n+5} = V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n$ $(n \geq 0)$ has a base of length 312 $(= 39 \times 8)$ with respect to the function ψ, and it is the following:

$$[1,1,1,3,6,3,5,9,8,4,2,1,6,3,7,1,9,8,1,8,9,8,7,6,2,5,$$

$$1,3,8,1,9,4,7,2,5,9,9,5,3,4,3,6,3,1,8,3,3,9,6,2,5,7,$$

$$2,4,2,2,8,9,7,1,9,7,6,3,8,6,3,8,1,8,8,1,8,8,6,4,9,8,$$

$$8,8,1,7,5,2,5,2,3,8,2,2,8,5,7,6,1,9,1,6,5,4,7,5,9,3,$$

$$1,7,7,9,9,6,2,6,5,1,2,7,3,9,4,7,3,8,4,8,3,8,4,9,5,2,$$

$$1,3,2,4,3,4,7,2,2,9,6,8,9,7,3,6,6,4,8,9,6,6,6,8,8,7,$$

$$8,1,5,2,5,3,7,4,3,4,3,3,8,3,3,2,1,8,8,4,5,8,6,4,9,5,$$

$$5,2,7,1,2,8,2,2,6,2,2,5,8,5,4,6,1,6,4,3,2,7,4,2,9,6,$$

$$1,4,4,6,3,9,8,3,2,7,2,4,9,6,1,4,6,8,7,8,6,8,1,3,8,8,$$

$$1,3,5,7,6,4,7,2,8,9,3,2,6,1,3,6,9,7,8,6,9,3,6,5,2,7,$$

$$5,7,8,2,2,6,7,7,6,1,9,3,8,9,3,5,1,8,8,7,2,8,6,4,9,2,$$

$$2,5,4,4,8,5,8,2,9,5,2,8,8,5,1,6,1,3,7,9,8,1,1,8,9,9]$$

c) $V_1 = 0, V_2 = V_3 = V_4 = V_5 = 1, V_{n+5} = V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n$ $(n \geq 0)$ has a base of length 312 $(= 39 \times 8)$ with respect to the function ψ, and it is the following:

$$[1,1,1,1,4,8,6,2,3,5,6,4,2,2,1,6,6,8,5,8,6,6,6,4,3,7,$$

$$8,1,5,6,9,2,5,9,4,2,4,6,7,5,6,1,7,8,9,4,2,3,8,8,7,1,$$

$$9,6,4,9,2,3,6,6,8,7,3,3,9,3,7,7,2,1,2,1,4,1,9,8,5,9,$$

5, 9, 9, 1, 6, 3, 1, 2, 4, 7, 8, 4, 7, 3, 2, 6, 4, 4, 1, 8, 5, 4, 4, 4, 7, 6,

7, 1, 7, 1, 4, 2, 6, 2, 6, 2, 9, 7, 8, 5, 4, 6, 3, 8, 8, 2, 9, 3, 3, 7, 6, 1,

2, 1, 8, 9, 3, 5, 8, 6, 4, 8, 4, 3, 7, 8, 3, 7, 1, 8, 9, 1, 8, 9, 8, 8, 7, 4,

9, 9, 1, 3, 8, 3, 6, 3, 5, 7, 6, 9, 3, 3, 1, 4, 2, 4, 5, 7, 4, 4, 6, 8, 2, 6,

8, 3, 9, 1, 9, 3, 7, 2, 4, 7, 5, 7, 7, 3, 2, 6, 7, 7, 7, 2, 2, 7, 7, 7, 7, 3,

4, 1, 4, 1, 4, 5, 6, 2, 9, 8, 3, 1, 5, 8, 7, 6, 9, 8, 2, 5, 3, 9, 9, 1, 9, 4,

5, 1, 2, 3, 6, 8, 2, 3, 4, 5, 4, 9, 7, 2, 9, 4, 4, 8, 9, 7, 5, 6, 8, 8, 7, 7,

9, 3, 7, 6, 5, 3, 6, 9, 2, 7, 9, 6, 6, 3, 4, 1, 2, 7, 8, 4, 4, 7, 3, 8, 8, 3,

2, 6, 9, 1, 3, 3, 4, 2, 4, 7, 2, 1, 7, 3, 2, 6, 1, 1, 4, 5, 8, 1, 1, 1, 7, 9]

d) $V_1 = V_2 = V_3 = V_4 = V_5 = 1$, $V_{n+5} = V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n$ $(n \geq 0)$ has a base of length 312 (= 39 × 8) with respect to the function ψ, and it is the following:

[1, 1, 1, 1, 1, 5, 9, 8, 6, 2, 3, 1, 2, 5, 4, 6, 9, 8, 5, 5, 6, 6, 3, 7, 9, 4,

2, 7, 2, 6, 3, 2, 2, 6, 1, 5, 7, 3, 4, 2, 3, 1, 4, 5, 6, 1, 8, 6, 8, 2, 7, 4,

9, 3, 7, 3, 8, 3, 6, 9, 2, 1, 3, 3, 9, 9, 7, 4, 5, 7, 5, 1, 4, 4, 3, 8, 2, 3,

2, 9, 6, 4, 6, 9, 7, 5, 4, 4, 2, 4, 1, 6, 8, 3, 4, 4, 7, 8, 8, 4, 4, 4, 1, 3,

7, 1, 7, 1, 1, 8, 9, 8, 9, 8, 6, 4, 8, 8, 7, 6, 6, 8, 8, 8, 9, 3, 9, 1, 3, 7,

5, 7, 5, 9, 6, 5, 5, 3, 1, 2, 7, 9, 4, 5, 9, 7, 7, 5, 6, 7, 5, 3, 8, 2, 7, 7,

9, 6, 4, 6, 5, 3, 6, 6, 8, 1, 6, 9, 3, 9, 1, 1, 5, 1, 8, 7, 4, 7, 9, 8, 8, 9,

5, 3, 6, 4, 9, 9, 4, 5, 4, 4, 8, 7, 1, 6, 8, 3, 7, 7, 4, 2, 5, 7, 7, 7, 1, 9,

4, 1, 4, 1, 1, 2, 9, 8, 3, 5, 9, 7, 5, 2, 1, 6, 3, 8, 2, 2, 3, 9, 6, 4, 6, 1,

8, 7, 8, 3, 9, 8, 8, 9, 1, 8, 7, 6, 4, 8, 6, 4, 1, 5, 6, 4, 2, 9, 8, 2, 7, 1,

9, 9, 1, 9, 2, 3, 6, 3, 5, 1, 9, 6, 6, 9, 4, 7, 5, 4, 2, 4, 4, 1, 6, 8, 5, 6,

8, 6, 6, 4, 3, 9, 1, 5, 4, 4, 5, 1, 1, 6, 8, 3, 1, 1, 1, 5, 2, 1, 1, 1, 1, 6]

Theorem 6: The non-homogenous sequence $\{G_i\}_{i=0}^{\infty}$, for which

$$G_1 = G_2 = 1, G_{n+2} = G_{n+1} + G_n + 1 \ (n \geq 0)$$

has a base of length 24 with respect to the function ψ, and it is the following:

[1, 1, 3, 5, 9, 6, 7, 5, 4, 1, 6, 8, 6, 6, 4, 2, 7, 1, 9, 2, 3, 6, 1, 8].

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