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On One Remarkable Identity Involving Bernoulli Numbers

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Abstract

In the present short note we propose and prove some new identities involving Bernoulli numbers and binomial coefficients. One of this identities is the main result of the paper.

First we remind that Bernoulli numbers (see [1]) are denoted by $B_m, m = 0, 1...$ and are introduced by:

$$B_0 = 1;$$

$$\sum_{p=1}^{m} \binom{m+1}{p} B_p = 0, m \geq 1$$

For example we have

$$B_0 = 1, B_1 = \frac{-1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, B_{10} = \frac{5}{66},$$

etc. It is well-known that:

$$B_{2t+1} = 0, t \geq 1$$

Let us introduce a bivariable polynomial $f_k(x, y)$ by:

$$f_k(x, y) := \sum_{p=1}^{k} \binom{k+1}{p} B_p \sum_{t=1}^{p} (-1)^p \binom{p}{t} x^{k+1-t} y^t \qquad (1)$$

where $k \geq 1$ is an integer. Obviously $deg f_k(x, y) = k + 1$, where deg means the degree of the polynomial. In [2] the following important results are proved

Lemma. The polynomial $f_k(x, y)$ from (1) is a symmetric function with respect to x and y, i.e the equality:

$$f_k(x, y) = f_k(y, x) \qquad (2)$$

holds. Also $f_k(x,y)$ admits the representation:

$$f_k(x,y) = x^{k+1} + \sum_{p=1}^{k} \binom{k+1}{p} B_p (x-y)^p x^{k+1-p}$$

Corollary 1. For $\alpha = 1, 2..., k$ the identity:

$$\sum_{p=\alpha}^{k} \binom{k+1}{p}\binom{p}{\alpha} B_p = (-1)^{k+1} \sum_{p=k+1-\alpha}^{k} \binom{k+1}{p}\binom{p}{k+1-\alpha} B_p \tag{3}$$

holds.

Proof. From (1) the coefficient corresponding to $x^{k+1-\alpha}y^{\alpha}$ is equal to

$$(-1)^{\alpha} \sum_{p=\alpha}^{k} \binom{k+1}{p}\binom{p}{\alpha} B_p,$$

and the coefficient corresponding to $x^{\alpha}y^{k+1-\alpha}$ is equal to

$$(-1)^{k+1-\alpha} \sum_{p=k+1-\alpha}^{k} \binom{k+1}{p}\binom{p}{k+1-\alpha} B_p$$

Hence (3) is true, since (2) holds.

Using that:

$$\binom{k+1}{p}\binom{p}{\alpha} = \binom{k+1}{\alpha}\binom{k+1-\alpha}{p-\alpha};$$

$$\binom{k+1}{p}\binom{p}{k+1-\alpha} = \binom{k+1}{\alpha}\binom{\alpha}{k+1-p},$$

we obtain from (3) the identity:

$$\sum_{p=\alpha}^{k} \binom{k+1-\alpha}{p-\alpha} B_p = (-1)^{k+1} \sum_{p=k+1-\alpha}^{k} \binom{\alpha}{k+1-p} B_p \tag{4}$$

which is valid for all $\alpha = 1, 2, ..., k$.

Now we are ready to formulate and prove the Main Result of the Paper.

Theorem. For every two positive integers α and β the identity:

$$(-1)^{\beta}(\binom{\beta}{0} B_{\alpha} + \binom{\beta}{1} B_{\alpha+1} + ... + \binom{\beta}{\beta-1} B_{\alpha+\beta-1})$$

$$= (-1)^{\alpha}(\binom{\alpha}{0} B_{\beta} + \binom{\alpha}{1} B_{\beta+1} + ... + \binom{\alpha}{\alpha-1} B_{\alpha+\beta-1}) \tag{5}$$

holds.

Proof. Putting in (4) $k = \alpha + \beta - 1$ we obtain the identity:

$$\binom{\beta}{0}B_\alpha + \binom{\beta}{1}B_{\alpha+1} + \dots + \binom{\beta}{\beta-1}B_{\alpha+\beta-1}$$

$$= (-1)^{\alpha+\beta}\left(\binom{\alpha}{0}B_\beta + \binom{\alpha}{1}B_{\beta+1} + \dots + \binom{\alpha}{\alpha-1}B_{\alpha+\beta-1}\right) \qquad (6)$$

From (6) follows immediately the identity (5) and also the identity:

$$(-1)^{\alpha}\left(\binom{\beta}{0}B_\alpha + \binom{\beta}{1}B_{\alpha+1} + \dots + \binom{\beta}{\beta-1}B_{\alpha+\beta-1}\right)$$

$$= (-1)^{\beta}\left(\binom{\alpha}{0}B_\beta + \binom{\alpha}{1}B_{\beta+1} + \dots + \binom{\alpha}{\alpha-1}B_{\alpha+\beta-1}\right) \qquad (7)$$

and the Theorem is proved.

Finally we need the following

Remark. If we agree that $\binom{0}{m} = 0$, for $m \geq 0$, and $\binom{0}{-1} = 0$ then, (5),(6) and (7) remain valid for each of the cases $\alpha = 0, \beta \neq 0$ and $\alpha \neq 0, \beta = 0$, because of the relation which defines Bernoulli numbers for $m \geq 1$. Moreover, (5),(6) and (7) will remain valid and for the case $\alpha = 0, \beta = 0$. But then these identities are trivial.

The above remark means that each one of (5),(6) and (7) is a strong generalization of the defining property for Bernoulli numbers

$$\sum_{p=0}^{m}\binom{m+1}{p}B_p = 0, m \geq 1$$

and for some other known identities involving both Bernoulli numbers and binomial coefficients.

References

[1] Borevich Z., Shafarevich I., *Number Theory (Second edition)*, Moscow, Nauka, 1972, (in Russian)

[2] Vassilev P., Vassilev-Missana M., *On the Sum of Equal Powers of the First n Terms of an Arithmetic Progression*, Notes on Number Theory and Discrete Mathematics, current issue: 15-21.