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On The Sum of Equal Powers of the First n Terms of an Arbitrary Arithmetic Progression (I)

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1 Introduction

In the paper an explicit formula for the sum

$$S_k(n) := a_1^k + a_2^k + \ldots + a_n^k \tag{1}$$

is proposed, where $n \geq 2$ and $k \geq 1$ are integers and the sequence $a_1, a_2, \ldots$ is an arbitrary arithmetic progression. An interesting fact is that this formula has the form:

$$S_k(n) = \frac{f_k(a_{n+1}, a_n) - f_k(a_1, a_0)}{(k+1)d}, \tag{2}$$

where: d is the difference of the progression; $a_0 := a_1 - d$ and $f_k(x, y)$ is a symmetric polynomial of two variables. The degree of this polynomial is equal to $k+1$. The coefficients of $f_k(x, y)$ depend on the classical Bernoulli numbers and on the binomial coefficients. The case $k = 1$ provides a new formula for the sum of the first n terms of an arbitrary arithmetic progression.

Another interesting fact is that the right hand-side of (2) admits a represantation in symmetric form with respect to $x = a_{n+1}, y = a_n$, after the introduction of an appropriate symbol.

2 Definition and Some Useful Properties of Bernoulli Numbers and Bernoulli Polynomials

Bernoulli numbers are denoted by $B_n, n = 0, 1, 2, \ldots$ and are introduced by:

$$B_0 = 1 \tag{3}$$

and

$$\sum_{p=0}^{m}\binom{m+1}{p}B_p = 0, m = 1, 2, ... \tag{4}$$

Using (3) and (4) one may calculate that:

$$B_1 = \frac{-1}{2}; B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, B_{10} = \frac{5}{66}, ... \tag{5}$$

It is a well-known fact that:

$$B_{2t+1} = 0, t \geq 1 \tag{6}$$

The relation (4) admits the *umbral calculus* representation:

$$(1+B)^{m+1} - B^{m+1} = 0, m = 1, 2, 3...$$

Bernoulli polynomials are introduced by:

$$B_m(z) := \sum_{p=0}^{m}\binom{m}{p}B_p z^{m-p}, \tag{7}$$

or in *umbral calculus* form by

$$B_m(z) := (z+B)^m, m \geq 1$$

Obviously we have

$$B_m(0) = B_m, m \geq 1$$

It is known that the relation:

$$B_m(z) = (-1)^m B_m(1-z), m \geq 1 \tag{8}$$

holds. The *umbral calculus* form of this relation is

$$(z+B)^m = (z-1-B)^m, m \geq 1 \tag{9}$$

All of the facts mentioned above about Bernoulli numbers and Bernoulli polynomials may be found in [1], [2], [3].

3 The Polynomial $f_k(x, y)$ and Some of Its Properties

The polynomial $f_k(x, y)$ is introduced by

$$f_k(x, y) := \sum_{p=1}^{k}\binom{k+1}{p}B_p \sum_{t=1}^{p}(-1)^t\binom{p}{t}x^{k+1-t}y^t, \tag{10}$$