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Remarks on Compositions of Numbers into Relatively Prime Parts

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The paper [2] introduced the study of the number-theoretic function $R_k(n)$ which counts the number of compositions of n into exactly k relatively prime positive parts. We state this definition as

$$R_k(n) = \sum_{\substack{a_1+a_2+a_3+\cdots a_k = n \\ (a_1, a_2, a_3, \ldots, a_k) = 1}} 1 . \qquad (1)$$

In that paper it was shown that the binomial coefficients may be expressed as a linear combination of bracket functions and in fact

$$\binom{n}{k} - \left[\frac{n}{k}\right] = \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j) . \qquad (2)$$

It was shown that when $k \geq 2$, then

$$R_k(n) \equiv 0 \pmod{k} \qquad (3)$$

for all natural numbers $n \geq k+1$ if and only if k is a prime number. Many other interesting results were given in [2], for example from (2) and (3) a necessary and sufficient condition for $k \geq 2$ to be prime is that

$$\binom{n}{k} \equiv \left[\frac{n}{k}\right] \pmod{k}$$

for every natural number n.

It was shown that

$$R_k(n) = \sum_{d|n} \binom{d-1}{k-1} \mu(n/d), \qquad (4)$$

where μ is the Möbius function.

Here $\binom{n-1}{k-1}$ is the number of compositions of n into k positive summands and the Möbius function acts to select those that have relatively prime parts.

The total number T(n) of compositions of n into relatively prime parts at all is, of course, given by

$$T(n) = \sum_{k=1}^{n} R_k(n) . \qquad (5)$$

The first few values of T(n) are: 1, 1, 3, 6, 15, 27, 63, 120, 252, 495, 1023, 2010, 4095, ... This sequence has been tabulated in Sloane [6].

It was also shown that

$$\sum_{k=1}^{n} R_k(n)\; x^{k-1} = \sum_{d|n} \mu(n/d)\, (x+1)^{d-1} . \qquad (6)$$

We mentioned in [2] that

$$\sum_{d|n} \mu(n/d)\, a^{d-1} \equiv 0 \pmod{n}$$

for all integers $a \geq 1$.

We found in [2] from a result of Gegenbauer that

$$a \sum_{d|n} \mu(n/d)\,(a+1)^{d-1} \equiv 0 \pmod{n} \qquad (7)$$

whenever $a \geq 1$, $n \geq 1$.

Now let us consider (6) when $x = 1$. We have

$$T(n) = \sum_{d|n} \mu(n/d)\, 2^{d-1}\ . \qquad (8)$$

We shall use this to announce the small

Theorem 1. $T(n) \equiv 0 \pmod{3}$, whenever $n \geq 3$. (9)

In other words the number of compositions of n into relatively prime parts is always a multiple of 3 when $n \geq 3$.

Remark. This solves a problem posed by Emeric Deutsch [1].

To prove this we will use a fact that may not be as well known as it should be. We have:

Lemma 1.

$$\sum_{d|n} \mu(n/d)\,(-1)^d = \begin{cases} -1 & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3 \end{cases}. \qquad (10)$$

Proof. Recall from the theory of Dirichlet series that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}\ , \qquad (11)$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = (2^{1-s} - 1)\zeta(s). \qquad (12)$$

Then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -1 + \frac{2}{2^s} \quad , \tag{13}$$

but

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(n/d)\,(-1)^d \quad , \tag{14}$$

so that our claim (9) follows by equating coefficients of $1/n^s$ in (13) and (14).

We now prove Theorem 1 as follows. We have

$$T(n) = \sum_{d|n} \mu(n/d)\, 2^{d-1} = \sum_{d|n} \mu(n/d)\,(3-1)^{d-1}$$

$$= \sum_{d|n} \mu(n/d) \sum_{i=0}^{d-1} \binom{d-1}{i} 3^i\,(-1)^{d-1-i}$$

$$= \sum_{d|n} \mu(n/d) \sum_{i=1}^{d} \binom{d-1}{i-1} 3^{i-1}(-1)^{d-i}$$

$$= \sum_{d|n} \mu(n/d) \sum_{i=1}^{n} \binom{d-1}{i-1} 3^{i-1}(-1)^{d-i}$$

$$= \sum_{i=1}^{n} 3^{i-1} \sum_{d|n} \mu(n/d) \binom{d-1}{i-1} (-1)^{d-i}$$

$$= \sum_{d|n} \mu(n/d)\,(-1)^{d-1} + \sum_{i=2}^{n} 3^{i-1} \sum_{d|n} \mu(n/d) \binom{d-1}{i-1} (-1)^{d-i},$$

Thus when $n \geq 3$, by (10) the first sum is congruent to $0 \pmod 3$, and the second summation is clearly a multiple of 3, so that our theorem follows. Q. E. D.

We wish next to observe that our number-theoretical function $R_k(n)$ affords a novel representation of the Möbius function that is not widely known. We have

Theorem 2.

$$\mu(n) = \sum_{k=1}^{n} R_k(n)\,(-1)^{k-1} . \tag{15}$$

Proof. This is a trivial consequence of (6) when we set $x = -1$.

Since there have been very few studies of $R_k(n)$, this representation does not appear in any standard text on number theory of which we know.

Other interesting results concerning $R_k(n)$ have been found by Shonhiwa [4], [5].

References

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