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NOTE ON SOME IDENTITIES RELATED TO BINOMIAL COEFFICIENTS

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The aim of the paper is the establishment of the following identities:

$$L^n = \sum_{k=0}^{[\frac{(L-1).n}{L}]} (-1)^k \binom{n}{k} \binom{L.(n-k)}{n}; \tag{1}$$

$$\sum_{k=0}^{[\frac{n}{3}]} (-1)^k \binom{n}{k} \binom{3(n-k)}{n} = \sum_{k=[\frac{n+1}{2}]}^{n} \binom{n}{k} \binom{k}{n-k} .3^k, \tag{2}$$

where $[x]$ denotes the integer part of x; L and n are natural numbers.

Let $\{.\}$ be any expansion involving the variable x. Then by $coef_{x^s}\{.\}$ we shall denote the coefficient preceeding x^s in this expansion.

Proof of (1): Using Newton's binomial formula we find

$$((1+x)^L - x^L)^n = \sum_{k\geq 0} (-1)^k \binom{n}{k} x^{L.k}(1+x)^{L.(n-k)}$$

$$= \sum_{k\geq 0} (-1)^k \binom{n}{k} x^{L.k} \sum_{j\geq 0} \binom{L.(n-k)}{j} x^j$$

$$= \sum_{k\geq 0} (-1)^k \binom{n}{k} . \sum_{j\geq 0} \binom{L.(n-k)}{j} x^{j+Lk}$$

Therefore,

$$coef_{x^{(L-1)n}}\{((1+x)^L - x^L)^n\} = \sum_{k\geq 0} (-1)^k \binom{n}{k} . \binom{L.(n-k)}{(L-1)n - Lk}$$

(since $j + Lk = (L-1).n$)

$$= \sum_{k\geq 0} (-1)^k \binom{n}{k} . \binom{L.(n-k)}{n}$$

(since $j \geq 0$ and $j = (L-1).n - L.k$ imply that $(L-1).n - L.k \geq 0$, i.e., $k \leq [\frac{(L-1).n}{L}]$)

$$= \sum_{k=0}^{[\frac{(L-1).n}{L}]} (-1)^k \binom{n}{k} . \binom{L.(n-k)}{n}.$$

Hence

$$coef_{x^{(L-1)n}}\{((1+x)^L - x^L)^n\} = \sum_{k=0}^{[\frac{(L-1).n}{L}]} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{n} \qquad (3)$$

On the other hand, we have

$$((1+x)^L - x^L)^n = (1 + \binom{L}{1} .x + \binom{L}{2} .x^2 + \ldots \binom{L}{L-1} .x^{L-1})^n.$$

Hence

$$coef_{x^{(L-1)n}}\{((1+x)^L - x^L)^n\} = \binom{L}{L-1}^n = L^n \qquad (4).$$

Now, (3) and (4) prove (1).

When $L = 2$, (1) yields the identity

$$2^n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2.(n-k)}{n}. \qquad (5)$$

The last identity is contained in [1], but in the form

$$2^n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \binom{2.(n-k)}{n-k} \qquad (6)$$

since

$$\binom{n}{n-k}\binom{2(n-k)}{n} = \binom{n-k}{k}\binom{2(n-k)}{n-k}.$$

The same way as in the proof of (1) we may establish that:

$$coef_{x^{Rn}}\{((1+x)^L - x^L)^n\} = \sum_{k=0}^{[\frac{Rn}{L}]} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{(L-R).n}, \qquad (7)$$

$$coef_{x^{Rn}}\{((1+x)^L - x^L)^n\} = coef_{x^{Rn}}\{1 + \binom{L}{1} .x + \binom{L}{2} .x^2 + \ldots \binom{L}{L-1} .x^{L-1})^n\}, \qquad (8)$$

where R is a natural number, such that $R \leq L$.

Proof of (2): Using Newton's binomial formula we find

$$((1+x)^L - x^L)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{s=0}^{L(n-k)} \binom{L(n-k)}{s} x^{s+L.k}.$$

Hence,

$$coef_{x^n}\{((1+x)^L - x^L)^n\} = \sum_{k=0}^{[\frac{n}{L}]} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{n}, \qquad (9)$$

since

$$\binom{L.(n-k)}{n-Lk} = \binom{L.(n-k)}{Ln-n} = \binom{L.(n-k)}{n}.$$

On the other hand we have

$$coef_{x^n}\{((1+x)^L - x^L)^n\} = coef_{x^n}\{(\sum_{k=0}^{L-1} \binom{L}{k} .x^k)^n\}. \tag{10}$$

Now, (9) and (10) imply

$$\sum_{k=0}^{[\frac{n}{L}]} (-1)^k \binom{n}{k} . \binom{L.(n-k)}{n} = coef_{x^n}\{(\sum_{k=0}^{L-1} \binom{L}{k} .x^k)^n\}. \tag{11}$$

Let $L = 3$. Then:

$$coef_{x^n}\{(\sum_{k=0}^{L-1} \binom{L}{k} .x^k)^n\} = coef_{x^n}\{(\sum_{k=0}^{2} \binom{3}{k} .x^k)^n\}$$

$$= coef_{x^n}\{(3x^2 + 3x + 1)^n\} = coef_{x^n}\{(3x^2 + (3x+1))^n\}$$

$$= coef_{x^n}\{\sum_{k=0}^{n} \binom{n}{k} .3^{n-k}.x^{2(n-k)}.(3x+1)^k\}$$

$$= coef_{x^n}\{\sum_{k=0}^{n} \binom{n}{k} .3^{n-k}.x^{2(n-k)}. \sum_{j=0}^{k} \binom{k}{j} .3^j.x^j\}$$

$$= coef_{x^n}\{\sum_{k=0}^{n} \binom{n}{k} .3^{n-k}. \sum_{j=0}^{k} \binom{k}{j} .3^j.x^{2(n-k)+j}\}.$$

Now, (11) (the case $L = 3$) and the last equality yield

$$\sum_{k=0}^{[\frac{n}{3}]} (-1)^k \binom{n}{k} . \binom{3.(n-k)}{n}$$

$$= coef_{x^n}\{\sum_{k=0}^{n} \binom{n}{k} .3^{n-k}. \sum_{j=0}^{k} \binom{k}{j} .3^j.x^{2(n-k)+j}\}. \tag{12}$$

Let $2(n-k) + j = n$. Then $j = 2k - n$ and since $j \geq 0$, it is fulfilled $2k - n \geq 0$, i.e., $k \geq [\frac{n+1}{2}]$.

On the other hand we have $k \leq n$. Hence

$$[\frac{n+1}{2}] \leq k \leq n. \tag{13}$$

For $j = 2k - n$ we have

$$\binom{k}{j} = \binom{k}{2k-n} = \binom{k}{n-k}.$$

Therefore, (12) and (13) yield

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \binom{n}{k} \cdot \binom{3.(n-k)}{n} = \sum_{k=[\frac{n+1}{2}]}^{n} \binom{n}{k} \cdot \binom{k}{n-k} .3^k$$

and (2) is proved.

REFERENCE:

[1] Riordan, J. Combinatorial Identities. John Wiley & Sons, New York, 1968.