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PRIME GRIDS IN THE MODULAR RING Z_6

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Abstract

Prime grids are set up in the Modular Ring Z_6 for the Classes $\bar{2}_6$ and $\bar{4}_6$. The regular formation of composites intrudes into the grid in a predictable manner, which indicates that the primes form in a structured rather than a haphazard manner when viewed in this way.

1. Introduction

In studying primes it is important to realise that these integers are class specific within modular residue classes [8]. For example, in the modular ring Z_6 [3], the characteristics of primes in $\bar{2}_6$ are different from those in $\bar{4}_6$. In this modular ring, odd numbers, *N*, with $3 \nmid N$ are given by $(6r_2 - 1) \in \bar{2}_6$ and $(6r_4 + 1) \in \bar{4}_6$ [3,4]. In Z_4, for instance, primes $(4r_1 + 1) \in \bar{1}_4$ equal a sum of squares while $(4r_3 + 3) \in \bar{3}_4$ do not [4,5].

Just as in the Sieve of Eratosthenes the production of primes follows a very ordered rule, so too in the grid of residue classes the production of primes follows a very ordered pattern complementary to the formation of the composites. In the grid for Z_6, the rows R_i, ($\bar{i}$ being the class) occupied by composites are well defined [6]: for $\bar{2}_6$,

$$R_2 = R' + pt, \quad t=0,1,2,3,\ldots, \tag{1.1}$$

with

$$R' = \tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2 + 1\right) + p\right), \; p \in \bar{2}_6$$

or

$$R' = \tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2+1\right)+2p\right),\ p \in \bar{4}_6$$

and for $\bar{4}_6$

$$R_4 = \tfrac{1}{6}\left(p^2-1\right)+pt, \tag{1.2}$$

in which p represents the lowest prime factor. Rows that do not fit these equations will be occupied by primes. There is a sense in which these modular rings are targeted rings for these corresponding numerical identities [2].

2. Primes in $\bar{2}_6$

A prime grid can be set up by taking p=1 in Equation (1.1) for $\bar{2}_6$ primes and Equation (1.2) for $\bar{4}_6$ primes. Thus for $\bar{2}_6$ primes (since $1 \in \bar{4}_6$)

$$\begin{aligned} p &= 6t_p + 5 \\ &= 6\left(t_p+1\right)-1 \end{aligned} \tag{2.1}$$

The subscript p on t is used to distinguish between the t values (representing grid slots) of the composite grid and those of the prime grid. Some examples of t_p are shown in Table1.

p t_p	5 0	11 1	17 2	23 3	29 4	41 6	47 7	53 8	59 9	71 11	83 13	89 14	
p t_p	101 16	107 17	113 18	131 21	137 22	149 24	167 27	173 28	179 29	191 31	197 32	227 37	233 38

Table 1: t_p values for primes in $\bar{2}_6$

The missing values of t_p in Table 1 indicate composite formation that absorbs these t_p slots. These composite t_p values are simply given by

$$t_p = A + pq,\ q=0,1,2,3,\ldots, \tag{2.2}$$

where p is the lowest prime factor.

Since from Equation (1.1)

$$t = \tfrac{1}{6p}\left(N-\left(p^2+ap\right)\right) \tag{2.3}$$

with N=pM, and a=2, $p \in \bar{2}_6$ and a=4, $p \in \bar{4}_6$, and since

$$N = 6t_p + 5,\ q = t,$$

we have

$$A = \tfrac{1}{6}\left(\left(p^2 + ap\right) - 5\right). \tag{2.4}$$

Obviously, for p=5, $t_p^* = 0{,}5, N^* = 5$ and is composite (the asterisks indicating the right end digit or RED). Some examples are given in Table 2.

t_p	12	23	36	122	124	644	715827882
N	779	143	221	737	749	3869	$2^{32}+1$
Factors	7×11	11×13	13×17	11×67	7×107	53×73	641×6700417
p	7	11	13	11	7	53	641
Class of p	$\bar{4}_6$	$\bar{2}_6$	$\bar{4}_6$	$\bar{2}_6$	$\bar{4}_6$	$\bar{2}_6$	$\bar{2}_6$
A	12	23	36	23	12	485	68693
q	0	0	0	9	16	3	1116629

Table 2: t_p slots occupied by composites

3. Primes in $\bar{4}_6$

For these primes, R'=0 in Equation (1.2), so that $R_4 = t_p$, and

$$p = 6t_p + 1.. \tag{3.1}$$

Some examples are in Table 3.

p	7	13	19	31	37	43	61	67
t_p	1	2	3	5	6	7	10	11
p	73	79	97	103	109	127	139	151
t_p	12	13	16	17	18	21	23	25
p	157	163	181	193	199	211	223	229
t_p	26	27	30	32	33	35	37	38
p	241	271	277	283	307	313	331	337
t_p	40	45	46	47	51	52	55	56

Table 3: t_p values for primes in $\bar{4}_6$

As for Class $\bar{2}_6$, the missing t_p slots in Table 3 have been replaced by composites (Table 4), and these composite-occupying slots are given by

$$t_p = \tfrac{1}{6}\left(p^2 - 1\right) + pt \text{ ,t=0,1,2,3,…,} \tag{3.2}$$

In which *p* is the lowest prime factor. When *t*=0, the composite is a square (which incidentally does not occur in $\bar{2}_6$). For $t_p^* = 4,9,\ p^* = 5,$ so that t_p slots with these REDs are inaccessible to the primes.

t_p	4	41	75	167	748	89478485
N	25	247	451	1003	4489	$2^{29} - 1$
Factors	5x5	13x19	11x41	17x59	67x67	233x1103x2089
p	5	13	11	17	67	233
$\tfrac{1}{6}\left(p^2 - 1\right)$	4	28	20	48	748	9048
t	0	1	3	7	0	383989

Table 4: t_p values for composites in $\bar{4}_6$

4. Final Comments

In number-theory analyses it is generally useful to keep the integer structure in mind. For example, the fact that certain classes do not contain even powers leads to a very simple proof of Fermat's assumption that only primes of the form $4r_1 + 1$ are a sum of squares, that is,

$$.\ p = x^2 + y^2. \tag{4.1}$$

For Z_4, with *x* even only Class $\bar{0}_4$ contains squares, and *y* is odd, so that y^2 is restricted to $\bar{1}_4$; that is,

$$.\ \bar{0}_4 + \bar{1}_4 = \bar{1}_4, \tag{4.2}$$

so that $\bar{3}_4$ $(4r_3+3)$ can never satisfy the requirements and there will never be an integer in that Class which is a sum of squares. On the other hand, the absence of even powers in $\bar{3}_4$ permits more space for primes. This accounts for the fact that in integer regions where these powers occur, the number of primes in $\bar{3}_4$ is greater than in $\bar{1}_4$.

We also know that Class $\bar{2}_4$ $(4r_2+2)$ contains no powers at all so that for $y=x^n$ with *x* even and *n*>1, $y \in \bar{0}_4 \subset Z_4$.

For Z_6 a somewhat similar situation occurs, integers divisible by 3 occur in $\bar{3}_6$ and $\bar{6}_6$, and no even powers occur in $\bar{5}_6$ so that for *n* even,

$$\begin{aligned} 2^n &\equiv \bar{1}_6 \\ &= (6r_1-2),\ n>1. \end{aligned} \tag{4.3}$$

Hence, with $1 \in \bar{4}_6$, then

$$\begin{aligned} 2^n+1 &\equiv \bar{1}_6+\bar{4}_6 \\ &= 6r_1-2+6r_4+1 \\ &= 6(r_1+r_4)-1 \\ &= 6r_2-1. \end{aligned} \tag{4.4}$$

Thus Fermat's numbers must always fall in $\bar{2}_6$[7]. On the other hand, if *n* is odd, $2^n+1 \in \bar{6}_6$ where 3|*N*, so that no primes can be formed.

When viewed from the perspective of integer structure, the primes do follow patterns interspersed by the orderly array of composites. In this context, the reader is referred to the lovely chapter entitled "The Primacy of Primes" in Conway and Guy's classic [1].

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