NNTDM 11 (2005) 1, 20-22

GOLDBACH'S n-PERFECT NUMBERS AS A KEY FOR PROVING THE GOLDBACH'S CONJECTURE

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The classical famous Goldbach's conjecture ($\mathcal{G}$) states:

$\mathcal{G}$: Every even number $m > 4$ is a sum of two odd primes.

For example: $6 = 3 + 3$, $8 = 3 + 5$, $10 = 3 + 7 = 5 + 5$, etc.

Below, we consider a modification of $\mathcal{G}$, that we call Strong Goldbach's conjecture $\mathcal{SG}$ (see also [1]):

$\mathcal{SG}$: Every even number $m \geq 8$ is a sum of two distinct odd primes.

Of course, $\mathcal{SG}$ implies $\mathcal{G}$, i.e., if $\mathcal{SG}$ is true then $\mathcal{G}$ is true, too.

For an integer $n \geq 1$ let $\sigma(n)$ denote (as usually) the sum of all divisors on n. It is well known that σ is a multiplicative function and $\sigma(1) = 1$. Also, function σ is used for introducing the so-called perfect numbers, i.e., the numbers for which

$$\sigma(n) = 2n$$

(see [2]).

We need the following modification of the concept of perfect number (for other modifications see [3]).

Definition. For an integer $n \geq 4$ we call an integer k Goldbach's n-perfect number if k satisfies the double inequality

$$1 \leq k \leq n - 3 \tag{1}$$

and the equality

$$\sigma(n^2 - k^2) = (n + 1)^2 - k^2 \tag{2}$$

holds.

For example, 1 is Goldbach's 4-perfect number, 2 is Goldbach's 5-perfect number, 3 is Goldbach's 6-perfect number, 4 is Goldbach's 7-perfect number, 3 and 5 are Goldbach's 8-perfect numbers, etc. As we see, sometimes it is possible for number k to be n-perfect number for different numbers n.

Let us consider the following conjecture

$\mathcal{H}$: For every integer $n \geq 4$ there exists at least one Goldbach's n-perfect number.

In this paper we will show that conjectures $\mathcal{SG}$ and $\mathcal{H}$ are equivalent to each other. Therefore, proving of $\mathcal{H}$ is a key for proving of $\mathcal{SG}$ and moreover - for proving of $\mathcal{G}$.

First, we need of

Lemma. Let $p > 1$ and $q > 1$ be different integers. Then the equality

$$\sigma(p.q) = (p+1).(q+1) \tag{3}$$

holds if and only if p and q are both primes.

Proof. Let $p > 1$ and $q > 1$ be different primes. We have

$$\sigma(p) = p + 1,$$

$$\sigma(q) = q + 1.$$

Then using that σ is a multiplicative function, we obtain

$$\sigma(p.q) = \sigma(p).\sigma(q) = (p+1).(q+1)$$

and (3) is proved.

Let $p > 1$ and $q > 1$ be different integers for which (3) holds. Then

$$\sigma(p.q) = 1 + p + q + pq.$$

The above equality means that all divisors of the product $p.q$ are numbers $1, p, q, p.q$. But the last is possible only in the case when p and q are both primes. The Lemma is proved.

Now, we are ready to prove the following

Theorem. $\mathcal{SG}$ is equivalent to $\mathcal{H}$.

Proof. First, we will prove that $\mathcal{SG}$ implies $\mathcal{H}$.

Let $n \geq 4$ be an arbitrary integer. Hence, $m = 2n$ is an even number and $m \geq 8$. Therefore,

$$m = 2n = p + q, \tag{4}$$

where p and q are distinct odd primes, since $\mathcal{SG}$ holds. Then, there exists an integer k satisfying (1) such that

$$p = n + k,\ q = n - k. \tag{5}$$

According to the Lemma, p and q from (5) satisfy (3), since they are distinct primes. Putting (5) in (3) we obtain

$$\sigma(n^2 - k^2) = (n + k + 1).(n - k + 1) = (n+1)^2 - k^2$$

and (2) is proved. Then, according to the definition, k is Goldbach's n-perfect number. Thus, we proved that $\mathcal{SG}$ implies $\mathcal{H}$.

Second, we will prove that $\mathcal{H}$ implies $\mathcal{SG}$.

Let $m \geq 8$ be an arbitrary even number. Then $m = 2n$ for some natural number $n \geq 4$. Therefore, for this n there is at least one Goldbach's n-perfect number k, since $\mathcal{H}$ is true. Therefore, k satisfies (1) and (2). Let $p > 1$ and $q > 1$ be given by (5). Then we rewrite (2) in the form (3) and note that p and q are distinct integers, since $p > q$. From the Lemma we conclude that p and q are both distinct odd primes. Finally, from (5) we obtain

$$p + q = (n + k) + (n - k) = 2n = m.$$

This proves that $\mathcal{H}$ implies $\mathcal{SG}$. The Theorem is proved.

Let for every even number $m \geq 8$ we put $m = 2n$ and denote by $R(n)$ the number of all different couples (p, q) of distinct odd primes satisfying (4) (according to $\mathcal{SG}$). Then it is easy to see that $R(n)$ coincides with the number of all different Goldbach's n-perferct numbers (according to $\mathcal{H}$).

References

[1] Zumkeller, R. On-Line Encyclopedia of Integer Sequences (N. Sloane, Ed.), http://www.research.att.com/ njas/sequences, A071681.

[2] Guy, R. Unsolved Problems in Number Theory, Springer, New York, 2004.

[3] Vassilev–Missana, M., K. Atanassov. Modifications of the concept of perfect numbers. Proceedings of Thirty First Spring Conference of the Union of Bulgarian Mathematicians, Borovets, 3-6 april 2002, 221-224.