Exploring the q-Riemann Zeta function and q-Bernoulli polynomials

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Abstract In this paper we study that the q-Bernoulli polynomial, which were constructed by T.Kim, are analytic continued to $\beta_s(z)$. A new formula for the q-Riemann Zeta function $\zeta_q(s)$ due to T.Kim (see [1,2,8]) in terms of nested series of $\zeta_q(n)$ is derived. The new concept of dynamics of the zeros of analytic continued polynomials is introduced, and an investing phenomenon of 'scattering' of the zeros of $\beta_s(z)$ is observed. Following the idea of q-zeta function due to T.Kim, we are going to use "Mathematica" to explore a formula for $\zeta_q(n)$

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Key words- q-Bernoulli polynomial, q-Riemann Zeta function

1. Introduction

Throughout this paper, $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ will denote the ring of integers, the field of real numbers and the complex numbers, respectively.

When one talks of q-extension, q is variously considered as an indeterminate, a complex numbers or p-adic numbers. In complex number field, we will assume that $|q| < 1$ or $|q| > 1$. The q-symbol $[x]_q$ denotes $[x]_q = \frac{1-q^x}{1-q}$.

In this paper we study that the q-Bernoulli polynomials due to T.Kim (see [1,8]) are analytic continued to $\beta_s(z)$. By those results, we give a new formula for the q-Riemann zeta function due to T.Kim, cf. [1,5,7], and investigate the new concept of dynamics of the zeros of analytic continued polynomials. Also, we observe an interesting phenomenon of 'scattering' of the zeros of $\beta_s(z)$. Finally, we are going to use a software package called "Mathematica" to explore dynamics of the zeros from analytic continuation for q-zeta function due to T.Kim.

2. Generating q-Bernoulli polynomials and Numbers

For $h \in \mathbb{Z}$, the q-Bernoulli polynomials due to T.Kim were defined as

$$\sum_{n=0}^{\infty} \frac{\beta_n(x,h|q)}{n!} t^n = -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_q t} + (1-q)h \sum_{l=0}^{\infty} q^{lh} e^{[l+x]_q t}, \quad (1)$$

for $x, q \in \mathbb{C}$, cf. [1,7].

In the special case $x = 0$, $\beta_n(0,h|q) = \beta_n(h|q)$ are called q-Bernoulli numbers, cf. [1,2,3,4].

By (1), we easily see that

$$\beta_n(x,h|q) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{j+h}{[j+h]_q} q^{jx}, \quad cf. [7,8], \quad (2)$$

where $\binom{n}{j}$ is binomial coefficient.

In Eq. (1), it is easy to see that

$$q^h (q\beta(h|q)+1)^n - \beta_n(h|q) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $\beta^n(h|q)$ by $\beta_n(h|q)$.

By differentiating both sides with respect to t in Eq. (1), we have

$$\beta_m(h|q) = -m \sum_{n=0}^{\infty} q^{hn} [n]_q^{m-1} - (q-1)(m+h) \sum_{n=0}^{\infty} q^{hn} [n]_q^m.$$

Expanding the Eq.(1) as a series and matching the coefficients on the both sides gives

$$\beta_0(2|q) = \frac{2}{[2]_q}, \quad \beta_1(2|q) = \frac{2q+1}{[2]_q[3]_q}, \quad \beta_2(2|q) = \frac{2q^2}{[3]_q[4]_q},$$
$$\beta_3(2|q) = -\frac{q^2(q-1)(2[3]_q+q)}{[3]_q[4]_q[5]_q}, \cdots, \quad \beta_0(h|q) = \frac{h}{[h]_q},$$
$$\beta_1(h|q) = -\frac{(1+q+\cdots+q^{h-1})+q(1+q+\cdots+q^{h-2})+\cdots+q^{h-1}}{[h]_q[h+1]_q}, \cdots .$$

By Eq. (1), the q-Bernoulli polynomials can be written as

$$\beta_m(x,h|q) = \sum_{j=0}^{m} \binom{m}{j} [x]_q^{n-j} q^{jx} \beta_j(h|q). \tag{3}$$

In the case $h = 0$, $\beta_m(x, 0|q)$ will be symbolically written as $\beta_{m,q}(x)$. Let $G_q(x,t)$ be generating function of q-Bernoulli polynomials as follows:

$$G_q(x,t) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}.$$

Then we easily see that

$$G_q(x,t) = \frac{q-1}{\log q} e^{\frac{t}{1-q}} - t \sum_{n=0}^{\infty} q^{h+x} e^{[n+x]_q t}, \quad |t| < 1, \text{ cf. } [5,6,7,8]. \tag{4}$$

For $x = 0, \beta_{n,q} = \beta_{n,q}(0)$ will be called q-Bernoulli numbers.

By (4), we easily see that

$$\beta_{m,q}(n) - \beta_{m,q} = m \sum_{l=0}^{n-1} q^l [l]_q^{m-1}.$$

Thus, we have

$$\sum_{l=0}^{n-1} q^l [l]_q^{m-1} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} q^{nl} \beta_{l,q} [n]_q^{m-l} + \frac{1}{m}(1-q^{mn})\beta_{m,q}. \tag{5}$$

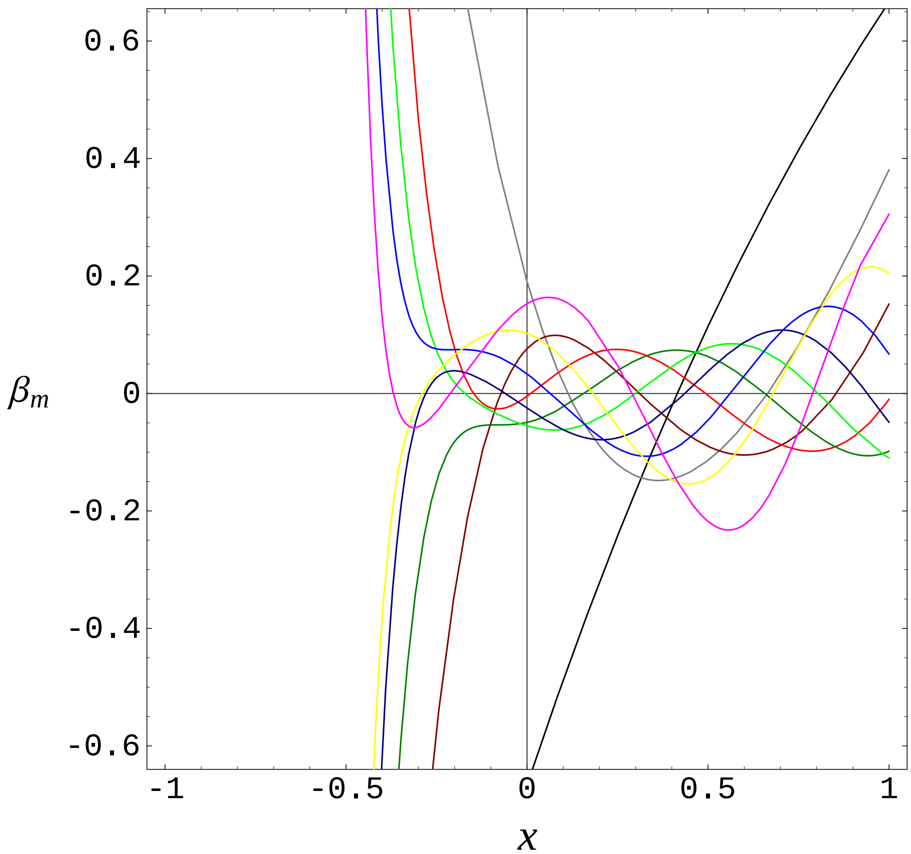

Figure 1: The curve of $\beta_m(x, 1|\frac{1}{2}), 1 \leq m \leq 10, -1 \leq x \leq 1$.

Zeros of q-Bernoulli polynomials are solutions of $\beta_m(x,1|q), x \in \mathbb{C}$.

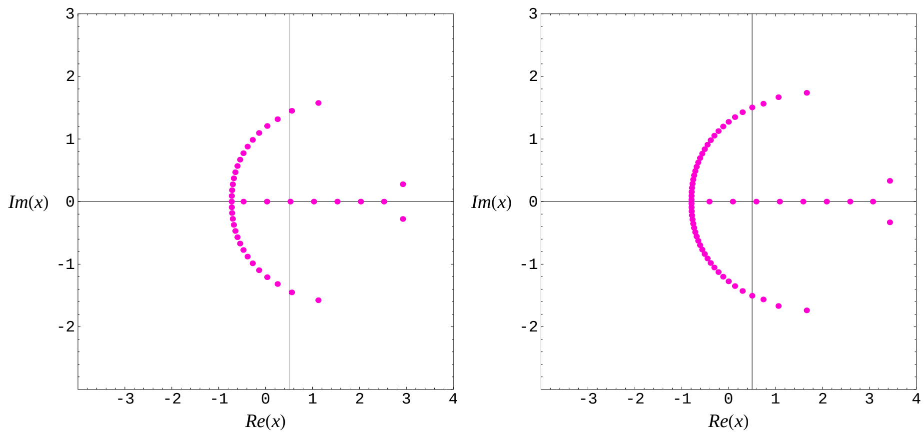

Figure 2: Zeros of q-Bernoulli polynomials $\beta_m(x,1|\frac{1}{2}), m = 40, 60$

Zeros of q-Bernoulli polynomials are solutions of $\beta_m(x,1|-q), x \in \mathbb{C}$.

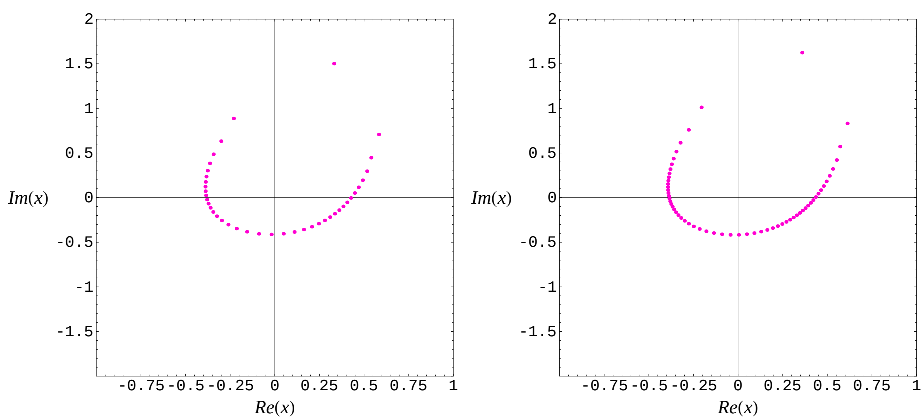

Figure 3: Zeros of q-Bernoulli polynomials $\beta_m(x,1|-\frac{1}{2}), m = 40, 60$

Zeros of q-Bernoulli polynomials are solutions of $\beta_m(x, 1|q), x \in \mathbb{C}$.

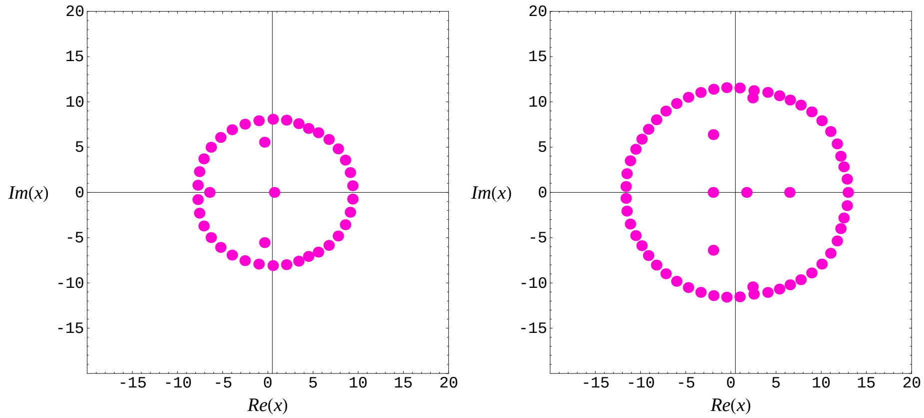

Figure 4: Zeros of q-Bernoulli polynomials $\beta_m(x, 1|\frac{11}{10}), m = 40, 60$

Zeros of q-Bernoulli polynomials are solutions of $\beta_m(x, 1| - q), x \in \mathbb{C}$.

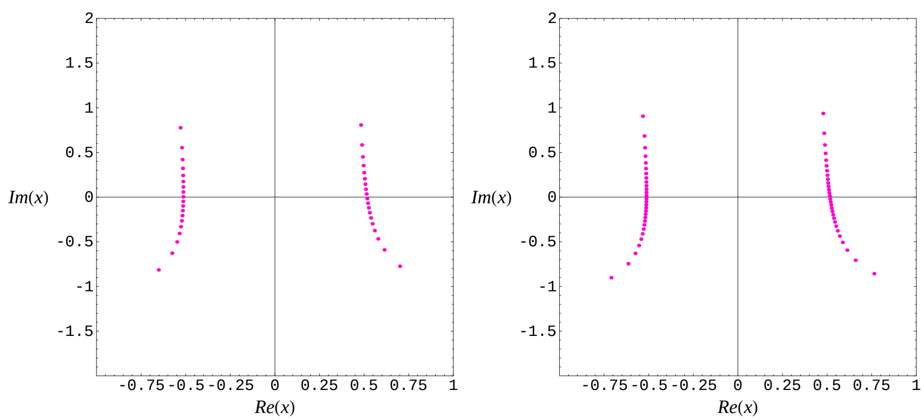

Figure 5: Zeros of q-Bernoulli polynomials $\beta_m(x, 1| - \frac{11}{10}), m = 40, 60$

Stacks of zeros of q-Bernoulli polynomials $\beta_n(x,1|q), 1 \le n \le 60$ from a 3-D structure.

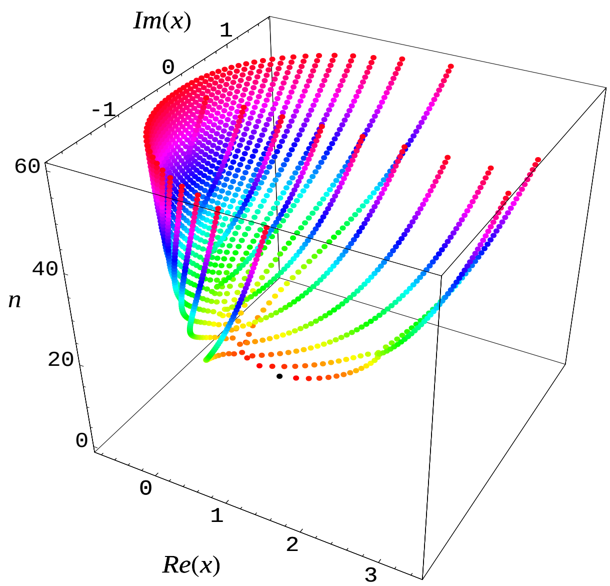

Figure 6: Stacks of zeros of q-Bernoulli polynomials $\beta_n(x,1|\frac{1}{2})$

3. q-Riemann zeta function

The q-Riemann zeta function due to T.Kim was defined as

$$\zeta_q^{(h)}(s) = \frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s-1}} + \sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s}}, \text{ for } s,h \in \mathbb{C}, \text{ cf. } [1,7]. \quad (6)$$

For $k \in \mathbb{N}, h \in \mathbb{Z}$, it was known that

$$\zeta_q^{(h)}(1-k) = -\frac{\beta_k(h|q)}{k}, \text{ cf. } [1,7]. \quad (7)$$

In the special case $h = s-1$, $\zeta_q^{(s-1)}(s)$ will be written as $\zeta_q(s)$. For $s \in \mathbb{C}$, we note that

$$\zeta_q(s) = \sum_{n=1}^{\infty}\frac{q^{n(s-1)}}{[n]_q^{s}}, \text{ cf. } [1,7]. \quad (8)$$

By (6), (7), and (8), we easily see that

$$\zeta_q(1-k) = -\frac{\beta_k(-k|q)}{k}, \text{ for } k \in \mathbb{N}, \text{ cf. } [5,6,7]. \tag{9}$$

From the above analytic continuation of q-Bernoulli numbers, we consider

$$\begin{aligned} &\beta_n = \beta_n(-n|q) \mapsto \beta(s), \\ &\zeta_q(-n) = -\frac{\beta_{n+1}(-n+1|q)}{n+1} \mapsto \zeta_q(-s) = -\frac{\beta(s+1)}{s+1} \\ &\Rightarrow \zeta_q(1-s) = -\frac{\zeta(s)}{s}. \end{aligned} \tag{10}$$

From the relation (10), we can define the other analytic continued half of q-Bernoulli numbers

$$\begin{aligned} &\beta(s) = -s\zeta_q(1-s), \quad \beta(-s) = s\zeta_q(1+s) \\ &\Rightarrow \beta_{-n} = \beta_{-n}(n|q) = \beta(-n) = n\zeta_q(n+1), n \in \mathbb{N}. \end{aligned} \tag{11}$$

The curve $\beta(s)$ runs through the points β_{-n} and $\lim_{n\to\infty} \beta_{-n} = n\zeta_q(n+1) = 0$.

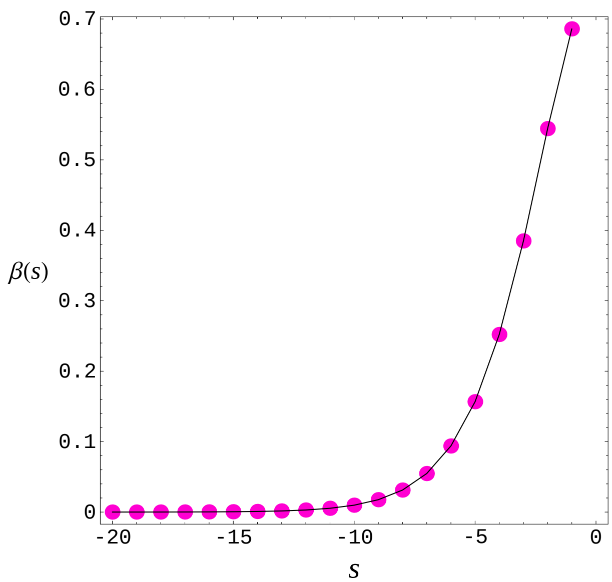

Figure 7: The curve $\beta(s)$ runs through the points $\beta_{-n}(n|\frac{1}{2})$

The curve of $\beta_{-n}(n|q)$ and $\lim_{n\to\infty}\beta_{-n} = n\zeta_q(n+1) = 0, q = 3/10, 5/10.$

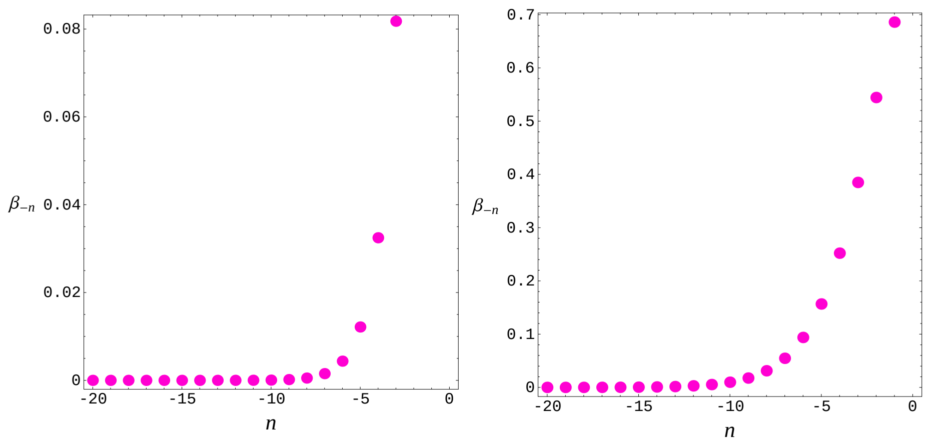

Figure 8: The curve of $\beta_{-n}(n|q)$

The curve of $\beta_{-n}(n|q)$ and $\lim_{n\to\infty}\beta_{-n} = n\zeta_q(n+1) = 0, q = 7/10, 9/10.$

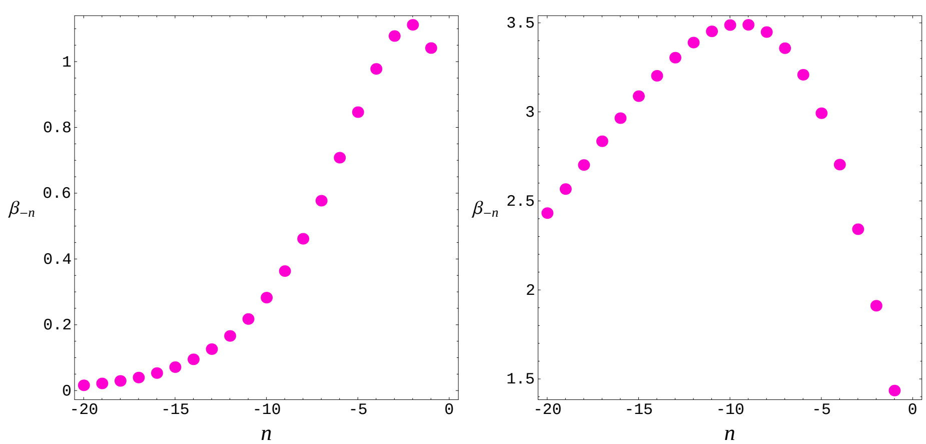

Figure 9: The curve of $\beta_{-n}(n|q)$

However, the curve $\beta_{-n}(n|q)$ grows $\sim n$ asymptotically as $q \to 1, (-n) \to -\infty$.

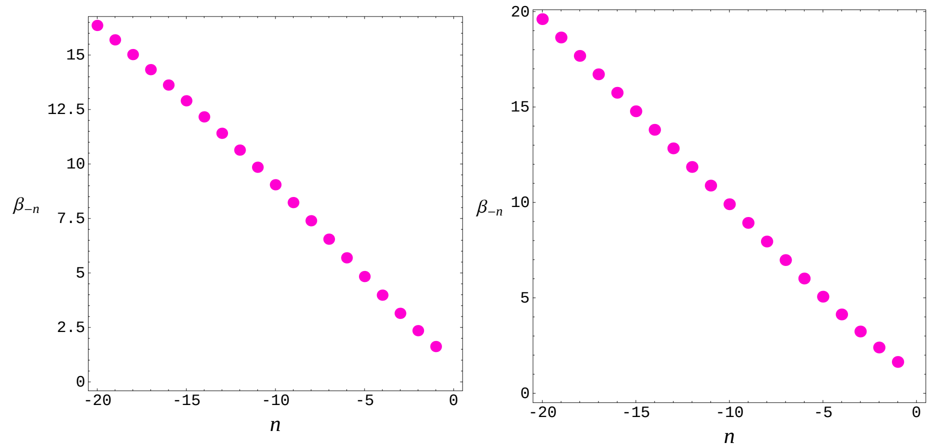

Figure 10: The curve of $\beta_{-n}(n|q), q = \frac{99}{100}, \frac{999}{1000}$

$$\zeta_q(m) = \sum_{n=1}^{\infty} \frac{q^{n(m-1)}}{[n]_q^m}, \Rightarrow \lim_{m \to \infty} \zeta_q(m) = 0.$$

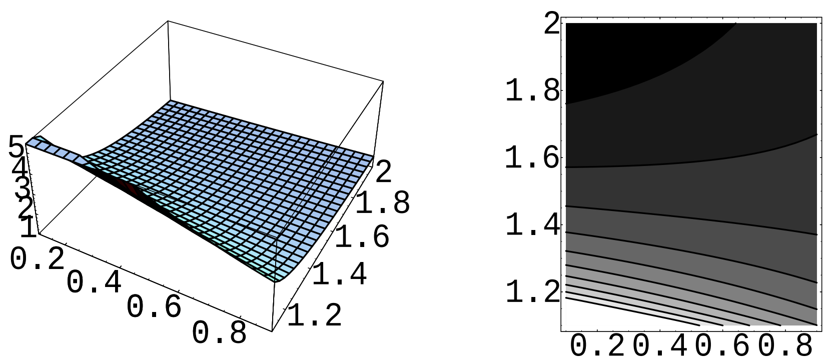

Figure 11: The plot of $\beta_q(s), 0.1 \leq s \leq 0.9, 1.1 \leq q \leq 2$

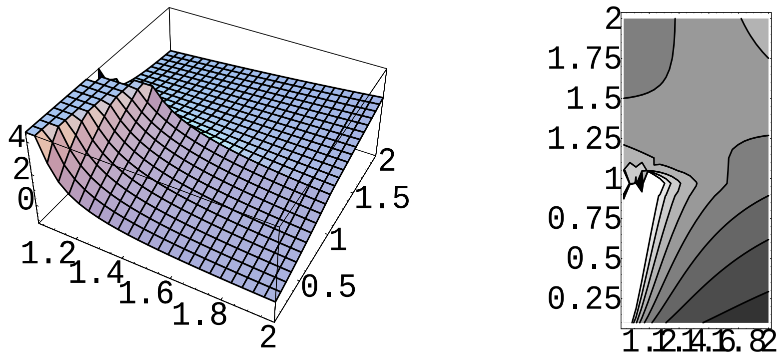

Figure 12: The plot of $\beta_q(s), 1.03 \leq s \leq 2, 0.1 \leq q \leq 2$

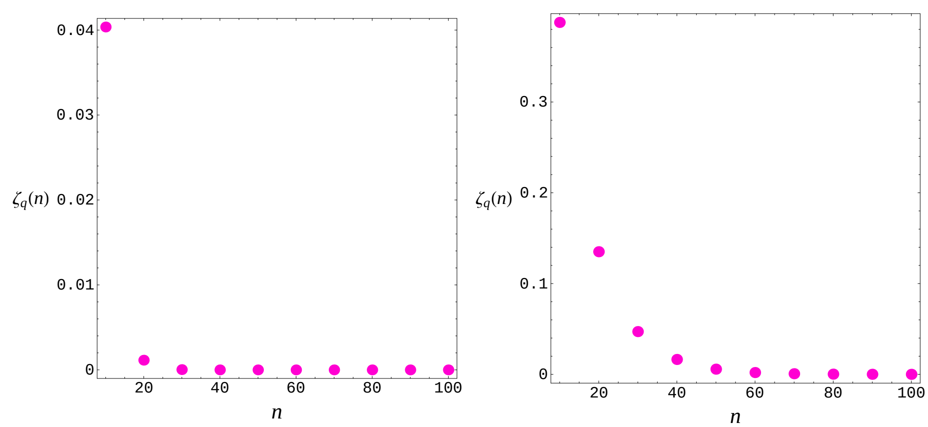

Figure 13: The curve of $\zeta_q(m), q = \frac{7}{10}, \frac{9}{10}$

4. Analytic continuation of q-Bernoulli polynomials

For consistency with the redefinition of $\beta_n = \beta(n)$ in (10) and (11),

$$\beta_n(x) = \beta_n(x, -n|q) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kx} [x]_q^{n-k}. \tag{12}$$

The analytic continuation can be then obtained as

$$n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C},$$

$$\beta_k \mapsto \beta(k + s - [s]|q) = -(k + (s - [s]))\zeta_q(1 - (k + (s - [s]))),$$

$$\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)}$$

$$\Rightarrow \beta_n(s) \mapsto \beta(s, w|q) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s)\beta(k + s - [s])q^{(k+s-[s])w}[w]_q^{[s]-k}}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)}$$

$$= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)\beta((k-1) + s - [s])q^{((k-1)+s-[s])w}[w]_q^{[s]+1-k}}{\Gamma(k + (s - [s]))\Gamma(2 + [s] - k)},$$

where $[s]$ gives the integer part of s, and so $s - [s]$ gives the fractional part.

Deformation of the curve $\beta(2, w)$ into the curve of $\beta(3, w)$ via the real analytic continuation $\beta(s, w), 2 \leq s \leq 3, -0.5 \leq w \leq 0.5$.

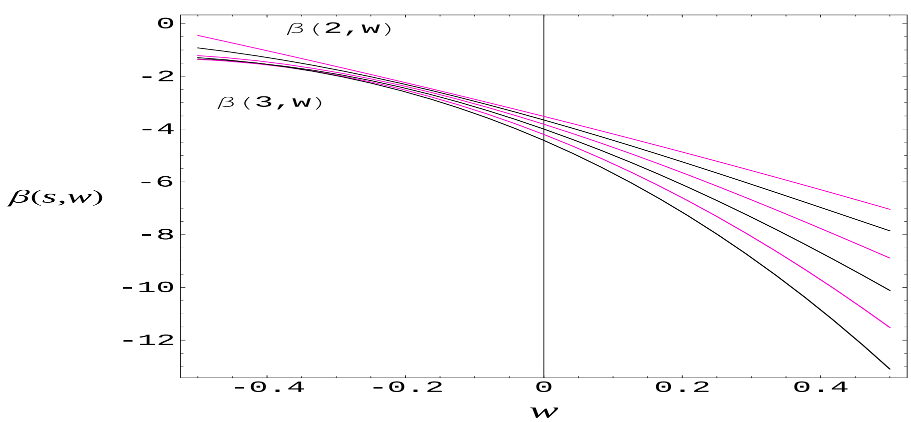

Figure 14: The curve of $\beta(s, w), 2 \leq s \leq 3, -0.5 \leq w \leq 0.5, q = \frac{11}{10}$.

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