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ON TWO SMARANDACHE'S PROBLEMS

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The 20-th problem from [1] is the following (see also Problem 25 from [3]):
Smarandache divisor products:

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19,$$

$$8000, 441, 484, 23, 331776, 125, 676, 729, 21952, 29, 810000, 31, 32768,$$

$$1089, 1156, 1225, 10077696, 37, 1444, 1521, 2560000, 41, ...$$

($P_d(n)$ is the product of all positive divisors of n.)

The 21-st problem from [1] is the following (see also Problem 26 from [3]:
Smarandache proper divisor products:

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1,$$

$$13824, 5, 26, 27, 784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39,$$

$$64000, 1, ...$$

($p_d(n)$ is the product of all positive divisors of n but n.)

Let us denote by $\tau(n)$ the number of all divisors of n. It is well-known (see, e.g., [2]) that

$$P_d(n) = \sqrt{n^{\tau(n)}} \tag{1}$$

and of course, we have

$$p_d(n) = \frac{P_d(n)}{n}. \tag{2}$$

But (1) is not a good formula for $P_d(n)$, because it depends on function τ and to express $\tau(n)$ we need the prime number factorization of n.

Below, following [4], we give other representations of $P_d(n)$ and $p_d(n)$, which do not use the prime number factorization of n.

Proposition 1.[4] For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{3}$$

holds.

Here and further the symbols

$$\prod_{k/n} \bullet \text{ and } \sum_{k/n} \bullet$$

mean the product and the sum, respectively, of all divisors of n and

$$\theta(n,k) \equiv [\frac{n}{k}] - [\frac{n-1}{k}] = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

The following assertion is obtained as a corollary of (2) and (3).

Proposition 2.[4] For $n \geq 1$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{5}$$

holds.

For $n = 1$ we have

$$p_d(1) = 1.$$

Proposition 3.[4] For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{6}$$

holds, where here and further we assume that $0! = 1$.

Now (2) and (6) yield.

Proposition 4.[4] For $n \geq 2$ representation

$$p_d(n) = \prod_{k=2}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{7}$$

holds.

Another type of representation of $p_d(n)$ is the following

Proposition 5.[4] For $n \geq 3$ representation

$$p_d(n) = \prod_{k=1}^{n-2} (k!)^{\theta(n,k)-\theta(n,k+1)}, \tag{8}$$

where $\theta(n,k)$ is given by (4).

Further, we need the following

Theorem.[4] For $n \geq 2$ the identity

$$\prod_{k=2}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n-1}(k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{9}$$

holds.

From (7), the left hand-side of (11) is equal to $p_d(n+1)$. From (8), the right side of (11) is equal to $p_d(n+1)$, too. Thetrefore, (11) is true.

Now, we shall deduce some formulae for

$$\prod_{k=1}^{n} P_d(k) \text{ and } \prod_{k=1}^{n} p_d(k).$$

Proposition 6. Let f be an arbitrary arithmetical function. Then the identity

$$\prod_{k=1}^{n}(P_d(k))^{f(k)} = \prod_{k=1}^{n} k^{\rho(n,k)} \tag{12}$$

holds, where

$$\rho(n,k) = \sum_{s=1}^{[\frac{n}{k}]} f(ks).$$

Proof. We use a well-known Dirichlet's identity

$$\sum_{k\leq n} f(k).\sum_{t/k} g(t) = \sum_{k\leq n} g(k).\sum_{s\leq \frac{n}{k}} f(ks),$$

where g is also arbitrary arithmetical function. Putting there $g(x) = \ln x$ for every real positive number x, we obtain (12), since

$$P_d(k) = \prod_{t/k} t.$$

When $f(x) \equiv 1$, as a corollary from (12) we obtain

Proposition 7. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} k^{[\frac{n}{k}]} \tag{13}$$

holds.

Now, we need the following

Lemma. For $n \geq 1$ the identity

$$\prod_{k=1}^{n}[\frac{n}{k}]! = \prod_{k=1}^{n} k^{[\frac{n}{k}]} \tag{14}$$

holds.

Proof. In the identity

$$\sum_{k\leq n} f(k). \sum_{s\leq \frac{n}{k}} g(s) = \sum_{k\leq n} g(k). \sum_{s\leq \frac{n}{k}} f(s),$$

that is valid for arbitrary two arithmetical functions f and g, we put:

$$g(x) \equiv 1,$$

$$f(x) = \ln x$$

for any positive real number x and (14) is proved.

From (13) and (14) we obtain

Proposition 8. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} [\frac{n}{k}]! \tag{15}$$

holds.

As a corollary from (2) and (15), we also obtain

Proposition 9. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=2}^{n} [\frac{n}{k}]! \tag{16}$$

holds.

Fom (9) and (16), we obtain

Proposition 10. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=1}^{n-1} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{17}$$

holds.

As a corollary from (17) we obtain, because of (2)

Proposition 11. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{18}$$

holds.

Now, we return to (12) and suppose that

$$f(k) > 0 \ (k = 1, 2, ...).$$

Then after some simple computations we obtain

Proposition 12. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{\sigma(n,k)} \tag{19}$$

holds, where

$$\sigma(n,k) = \frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)}.$$

For $n \geq 2$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{\sigma(n,k)} \tag{20}$$

holds.

Note that although f is an arbitrary arithmetical function, the situation with (19) and (20) is like the case $f(x) \equiv 1$, because

$$\frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks) - \sum_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)} = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

Finally, we use (12) to obtain some new inequalities, involving $P_d(k)$ and $p_d(k)$ for $k = 1, 2, ..., n$. Putting

$$F(n) = \sum_{k=1}^{n} f(k)$$

we rewrite (12) as

$$\prod_{k=1}^{n} (P_d(k))^{\frac{f(k)}{F(n)}} = \prod_{k=1}^{n} k^{\delta_1}, \tag{21}$$

where

$$\delta_1 = \frac{\sum_{s=1}^{[\frac{n}{k}]} f(ks)}{F(n)}.$$

Then we use the well-known Jensen's inequality

$$\sum_{k=1}^{n} \alpha_k x_k \geq \prod_{k=1}^{n} x_k^{\alpha_k},$$

that is valid for arbitrary positive numbers x_k, α_k $(k = 1, 2, ..., n)$ such that

$$\sum_{k=1}^{n} \alpha_k = 1,$$

for the case:

$$x_k = P_d(k),$$
$$\alpha_k = \frac{f(k)}{F(n)}.$$

Thus we obtain from (21) inequality

$$\sum_{k=1}^{n} f(k).P_d(k) \geq (\sum_{k=1}^{n} f(k)).\prod_{k=1}^{n} k^{\delta_2}, \tag{22}$$

where

$$\delta_2 = \frac{\sum\limits_{s=1}^{[\frac{n}{k}]} f(ks)}{\sum\limits_{s=1}^{n} f(s)}.$$

If $f(x) \equiv 1$ then (22) yields the inequality

$$\frac{\sum\limits_{k=1}^{n} P_d(k)}{n} \geq \prod_{k=1}^{n} (\sqrt[n]{k})^{[\frac{n}{k}]}. \tag{23}$$

If we put in (22)

$$f(k) = \frac{g(k)}{k}$$

for $k = 1, 2, ..., n$, then we obtain

$$\sum_{k=1}^{n} g(k).p_d(k) \geq (\sum_{k=1}^{n} \frac{g(k)}{k}).\prod_{k=1}^{n} (\sqrt[k]{k})^{\delta_3}, \tag{24}$$

where

$$\delta_3 = \frac{\sum\limits_{s=1}^{[\frac{n}{k}]} \frac{g(ks)}{s}}{\sum\limits_{s=1}^{n} \frac{g(s)}{s}},$$

because of (2).

Let $g(x) \equiv 1$. Then (24) yields the very interesting inequality

$$\left(\frac{\sum_{k=1}^{n} p_d(k)}{H_n}\right)^{H_n} \geq \prod_{k=1}^{n} (\sqrt[k]{k})^{H_{[\frac{n}{k}]}},$$

where H_m denotes the m-th partial sum of the harmonic series, i.e.,

$$H_m = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m}.$$

All of the above inequalities become equalities if and only if $n = 1$.

References

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