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# A FERMATIAN STAUDT-CLAUSEN THEOREM

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**ABSTRACT**
This paper looks at the Staudt-Clausen theorem within the framework of various generalization of the Bernoulli numbers. The historical background to the problem is reviewed, and a solution to a problem of Morgan Ward is put forward. Generalized Hurwitz series are utilised in the development of the results.

## 1. INTRODUCTION

Morgan Ward [21] once posed the problem whether a suitable definition for generalized Bernoulli numbers could be framed so that a generalized Staudt-Clausen Theorem might exist for them within the framework of the Fontené-Jackson calculus [6,8].

Carlitz [4] outlined a partial generalization of the Staudt-Clausen theorem with the Fontené-Jackson operators. The purpose of this paper is to show that Ward's problem can be solved with the adaptation of a method used by Carlitz for the ordinary Staudt-Clausen theorem [9] and for coefficients of more general series [11].

## 2. DEFINITIONS

To define these generalized Bernoulli numbers, $B_{n,q}$, we use the *n*th reduced Fermatians of index $q$ as defined in Shannon [17], namely:

$$\frac{t}{E_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q} t^n / \underline{q}_n!, \tag{2.1}$$

where

$$E_q(t) = \sum_{n=0}^{\infty} t^n / \underline{q}_n!, \tag{2.2}$$

and

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n < 0) \\ 1 + q + q^2 + \ldots + q^{n-1} & (n > 0) \\ 1 & (n = 0) \end{cases}$$

and

$$\underline{q}_n! = \underline{q}_n \underline{q}_{n-1}!,$$

so that

$$\underline{1}_n = 1, \ \underline{1}_n! = n!.$$

Thus,

$$B_{n,1} = B_n,$$

the ordinary Bernoulli numbers, $B_n$, for which a form of the Staudt-Clausen theorem can be stated as follows:

$$pB_n \equiv \begin{cases} -1 (\text{mod } p) & (p-1 \mid n) \\ 0 \ (\text{mod } p) & (p-1 \nmid n) \end{cases} \tag{2.3}$$

where $p$ is an odd prime (and hence $n$ is even). Our analogue of (2.3) for the Fermatian Bernoulli numbers is in (5.9).

## 3. GENERALIZED DIFFERENTIAL OPERATORS

Carlitz also studied generalized versions of differential operators in the form of Chak derivatives defined by

$$D_q(fx) = \frac{f(qx) - f(x)}{qx - x}. \tag{3.1}$$

He has also investigated properties associated with the Schur derivative

$$\Delta a_m = \left(a_{m+1} - a_m\right)/ p^{m+1}, \tag{3.2}$$

where $\{a_m\}$ is a sequence and $p$ is a prime number [7]. In the same spirit then it is convenient to define formally the Fermatian differential operator

$$D_{x,q} = \underline{q}_n x^{n-1}. \tag{3.3}$$

It follows that

$$D_{x,1} x^n = n x^{n-1}.$$

Some properties of the Fermatian differential operator follow if we define

$$D_{q,x} = \left(1 - q^n\right) x^{n-1}, \tag{3.4}$$

and

$$D_{q,x} q = 0,$$

so that

$$\left(1 - q\right) D_{x,q} x^n = (1 - q^n) x^{n-1} = D_{q,x} x^n. \tag{3.5}$$

Then

$$D_{x,q} a x^n = a D_{x,q} x^n,$$

where $a$ is a constant, and for $f(y)$, a function of $y$,

$$D_{x,q} f(y) = D_{y,q} f(y) D_{x,q} y,$$

which reduces to the ordinary ‘function of a function rule’ when $q$ is unity.

$$D_c f(y) = D_y f(y) D_x y$$

Other properties include

$$D_{xq} y^n = \underline{q}_n y^{n-1} D_{xq} y$$

and

$$D_{xq}\left(x^n + y^n\right) = D_{xq} x^n + D_{xq} y^n, \qquad (3.6)$$

and, for $u,v$, functions of $x$

$$D_{xq}^n uv = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} D_{xq}^r u D_{xq}^{n-r} v,$$

which is analogous to Leibnitz’ Theorem for the $n$th derivative of a product of two functions:

$$D^n uv = \sum_{r=0}^{n} \binom{n}{r} D^r u D^{n-r} v,$$

in which

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{\underline{q}_n!}{\underline{q}_r! \underline{q}_{n-r}!}.$$

The proof of (3.6) follows readily by induction on $n$. We also define the operator $I_{xq}$ formally by

$$I_{xq} D_{xq} f(x) = f(x). \qquad (3.7)$$

and

$$D_{xq} f(x) = I_{xq}^{-1} f(x).. \qquad (3.8)$$

Thus, for $n \neq -1$,

$$I_{xq} = \frac{1-q}{1-q^{n+1}} x^{n+1} + C$$
$$= \frac{x^{n+1}}{\underline{q}_{n+1}} + C,$$

in which C is a constant determined by the initial conditions, and, for $n=-1$, we have

$$. I_{xq} x^{-1} = L_q(x) + C$$

where

$$L_q(1+x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{r+1}}{q^r \underline{q}_{r+1}}.$$

is an analogue of the logarithmic function.

From (3.5) we have that

$$I_{xq}^{-1} f(x) = (1-q)^{-1} D_{qx} f(x), \quad .$$

and so we can introduce $I_{qx}$

$$I_{xq}^{-1} f(x) = (1-q)^{-1} I_{qx}^{-1} f(x)..$$

This means that

$$I_{qx} f(x) = D_{qx}^{-1} f(x).$$

and

$$I_{xq} f(x) = (1-q) I_{qx} f(x)..$$

When $q$=1,

$$I_{x1} x^n = \frac{x^{n+1}}{n+1} + C, \; n \neq -1,. \tag{3.9}$$

so that $I_{x1} x^n$ and $\int x^n dx$ differ by a constant only, which can be made zero with suitable limits; that is, the *I* operation is a generalization of integration. One can also define generalizations of the circular and hyperbolic functions in a somewhat similar manner [16].

## 4. GENERALIZED HURWITZ SERIES

We shall call series of the form

$$\sum_{n=0}^{\infty} a_n t^n / \underline{q}_n!, \tag{4.1}$$

where the $a_n$ are arbitrary integers, a generalized Hurwitz series (GH-series). When $q$ is unity we get an ordinary Hurwitz series [5]. If we consider another GH-series

$$\sum_{n=0}^{\infty} b_n t^n / \underline{q}_n!, \tag{4.2}$$

then the product of (4.1) and (4.2)

$$\sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} a_r b_{n-r} \right) t^n / \underline{q}_n!, \tag{4.3}$$

is also a GH-series.

The Fontené-Jackson type derivatives and integrals of GH-series are also GH-series, namely

$$D_{tq} \sum_{n=0}^{\infty} a_n t^n / \underline{q}_n! = \sum_{n=0}^{\infty} a_{n+1} t^n / \underline{q}_n!,$$

and

$$I_{xq}\left|\sum_{n=0}^{\infty} a_n x^n / \underline{q}_n!\right|_0^t = \sum_{n=1}^{\infty} a_{n-1} t^n / \underline{q}_n!.$$

For a series without constant term such as

$$H_1(t) = \sum_{n=1}^{\infty} a_n t^n / \underline{q}_n!, \tag{4.4}$$

it follows from

$$H_1^k(t)/\underline{q}_k! = I_{xq} D_{xq} H_1^k(x)/\underline{q}_k!\Big|_0^t$$
$$= I_{xq} H_1^{k-1}(x) D_{xq} H_1(x)/\underline{q}_{k-1}!\Big|_0^t$$

that $H_1^k(t)/\underline{q}_k!$ is a GH-series for all $k \ge 1$. This result can also be stated in the form

$$H_1^k(t) \equiv 0 \,(\mathrm{mod}\, \underline{q}_k!) \tag{4.5}$$

where by the statement

$$\sum_{n=0}^{\infty} a_n t^n / \underline{q}_n! \equiv \sum_{n=0}^{\infty} b_n t^n / \underline{q}_n! \;(\mathrm{mod}\, \underline{q}_m)$$

is meant that the system of congruences

$$a_n \equiv b_n \;\; (\mathrm{mod}\, \underline{q}_m), \;\; (n = 0,1,2,...,) \tag{4.5}$$

is satisfied. This is equivalent to the assertion

$$\sum_{n=0}^{\infty} a_n t^n / \underline{q}_n! = \sum_{n=0}^{\infty} b_n t^n / \underline{q}_n! + \underline{q}_m H(t) \tag{4.6}$$

where H(t) is some GH-series.
We now consider the class of GH-series

$$f(t) = \sum_{n=1}^{\infty} a_n t^n / \underline{q}_n!, \;\; (a_1 = 1) \tag{4.7}$$

and

$$D_{tq} = \sum_{n=0}^{\infty} A_n f^n(t), \;\; (A_0 = 1) \tag{4.8}$$

where the $A_n$ are integers. It follows from (4.6) and (4.8) that

$$D_{tq} f(t) \equiv \sum_{n=0}^{m-1} f^n(t) \;\; (\mathrm{mod}\, \underline{q}_m)$$

since $\underline{q}_m \mid \underline{q}_m! \mid A_k f^k(t)$, $k \geq m$. It follows from (4.7) that $f(0)=0$, and so

$$D_{0q} f(t) = 1$$

where $D_{oq}^r f(t)$ denotes the $r$th generalized Fontené-Jackson derivative of $f(t)$ evaluated at $t=0$. A result we shall use is

$$D_{0q}^m f^k(t) \equiv 0 \ (\text{mod } \underline{q}_m). \qquad (k>1). \qquad (4.9)$$

This arises because

$$\begin{aligned} D_{0q}^m f^{k+1}(t) &= D_{0q}^m f^k(t) f(t) \\ &\equiv f(0) D_{0q}^m f^k(t) + f^k(0) D_{0q}^m f(t) \ (\text{mod } \underline{q}_m). \end{aligned}$$

Another result to be used later is

$$D_{0q}^m f^m(t)/\underline{q}_m \equiv \underline{q}_{m-1}! \ (\text{mod } \underline{q}_m). \qquad (4.10)$$

*Proof*:

$$\begin{aligned} D_{0q}^m f^m(t)/\underline{q}_m &= D_{0q}^{m-1}\left(D_{0q} f^m(t)\right)/\underline{q}_m \\ &= D_{0q}^{m-1}\left(f^{m-1}(t) D_{0q} f(t)\right) \\ &\equiv D_{0q}^{m-1} f^{m-1}(t) \ (\text{mod } \underline{q}_m) \\ &\equiv D_{0q}^{m-2}\left(\underline{q}_{m-1} f^{m-2}(t) D_{0q} f(t)\right) (\text{mod } \underline{q}_m) \\ &\equiv \underline{q}_{m-1} D_{0q}^{m-2} f^{m-2}(t) \ (\text{mod } \underline{q}_m) \\ &\equiv \underline{q}_{m-1}! \ (\text{mod } \underline{q}_m). \end{aligned}$$

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## 5. FERMATIAN STAUDT-CLAUSEN THEOREM

If we put

$$f^{m-1}(t) = \sum_{n=m-1}^{\infty} a_n^{'} t^n / \underline{q}_n!, \qquad (5.1)$$

then, since $\underline{q}_m \mid \underline{q}_m! \mid f^m(t)$ from (4.5),

$$f^{m-1}(t) f(t) \equiv 0 \, (\text{mod } \underline{q}_m),$$

and we get

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} a_r a_{n-r}^{'} \equiv 0 \, (\text{mod } \underline{q}_m), \ n \geq \underline{q}_m. \qquad (5.2)$$

Note that (4.7) and (5.1) imply that

$$a'_{m-1} = \underline{q}_{m-1}!.$$

From the definition of $a_n$ and $a'_n$, for $n$=$m$=1, Congruence (5.2) reduces to

$$\underline{q}_{m+1} a_1 a'_m \equiv 0 (\bmod \underline{q}_m)$$

so that

$$a'_m \equiv 0 (\bmod \underline{q}_m) \qquad \text{(from [18]).}$$

For $n$=$m$+2, (5.2) becomes

$$\underline{q}_{m+2} a_1 a'_{m+1} + \begin{bmatrix} m+2 \\ 2 \end{bmatrix} a_2 a'_m \equiv 0 (\bmod \underline{q}_m)$$

which simplifies to

$$\underline{q}_{m+2} a_1 a'_{m+1} \equiv 0 (\bmod \underline{q}_m)$$

so that

$$a'_{m+1} \equiv 0 (\bmod \underline{q}_m)$$

since

$$\underline{q}_{m+2} \equiv 1 + q (\bmod \underline{q}_m) \qquad \text{(see [18])}$$

and

$$1 = \left(1 + q, \underline{q}_m\right) \qquad (q > 0, m > 2).$$

Continuing in this way, we get

$$a'_m \equiv a'_{m+1} \equiv \ldots \equiv a'_{2m-3} \equiv 0 (\bmod \underline{q}_m).$$

For $n$=2$m$-1, we get

$$\underline{q}_{2m-1} a_1 a'_{2m-2} + \ldots + \begin{bmatrix} 2m-1 \\ m \end{bmatrix} a_m a'_{m-1} \equiv 0 (\bmod \underline{q}_m),$$

which gives

$$\underline{q}_{m-1} a'_{2m-2} + a_m a'_{m-1} \equiv 0 (\bmod \underline{q}_m) \qquad \text{(from [18])}$$

so that

$$a'_{2m-2} \equiv q a_m a'_{m-1} \equiv q a_m \underline{q}_{m-1}! (\bmod \underline{q}_m).$$

From (5.1) we get

$$\begin{aligned} D_{tq}^{m-1} f^{m-1}(t) &= \sum_{n=m-1}^{\infty} a'_n t^{n-m+1} / \underline{q}_{n-m+1}! \\ &\equiv a'_{m-1} + a'_{2m-2} t^{m-1} / \underline{q}_{m-1}! + \ldots \\ &\equiv \underline{q}_{m-1}! + q a_m \left(a'_{m-1} t^{m-1} / \underline{q}_{m-1}! + \ldots\right) \\ &\equiv \underline{q}_{m-1}! + q a_m f^{m-1}(t) \ (\bmod \underline{q}_m). \end{aligned} \tag{5.3}$$

A solution of (5.3) is

$$f^{m-1}(t) \equiv \underline{q}_{m-1}! \sum_{n=1}^{\infty} (q a_m)^{n-1} \frac{t^{n(m-1)}}{\underline{q}_{n(m-1)}!} (\bmod \underline{q}_m). \tag{5.4}$$

This can be verified as follows

$$D_{0q}^{m-1} f^{m-1}(t) \equiv \underline{q}_{m-1}! \sum_{n=1}^{\infty} (qa_m)^{n-1} \frac{t^{(n-1)(m-1)}}{\underline{q}_{(n-1)(m-1)}!}$$

$$\equiv \underline{q}_{m-1}! \sum_{n=0}^{\infty} (qa_m)^{n} \frac{t^{n(m-1)}}{\underline{q}_{n(m-1)}!}$$

$$\equiv \underline{q}_{m-1}! + qa_m f^{m-1}(t) \ (\mathrm{mod}\, \underline{q}_m).$$

Now let

$$\lambda(t) = \sum_{n=1}^{\infty} e_n f^n(t) / \underline{q}_n! \qquad (e_1 = 1)$$

denote the inverse of $f(t)$, so that

$$t = \sum_{n=1}^{\infty} e_n f^n(t) / \underline{q}_n!. \tag{5.5}$$

Differentiating (5.5) we get

$$1 = \sum_{n=0}^{\infty} e_{n+1} \frac{f^n(t)}{\underline{q}_n!} D_{tq} f(t).$$

Comparison with (4.8) yields

$$e_{n+1} \equiv 0 \ (\mathrm{mod}\, \underline{q}_n!). \tag{5.6}$$

(5.5) can be re-written as

$$t / f(t) = \sum_{n=0}^{\infty} e_{n+1} f^n(t) / \underline{q}_{n+1}!. \tag{5.7}$$

It follows from (4.5) and (5.6) that $e_{n+1} f^n(t) / \underline{q}_{n+1}!$ is a GH-series and the coefficients of $f^n(t)$ are multiples of $\underline{q}_{n+1}!$. At this stage with ordinary integers one can proceed by using the fact that $n!$ is a multiple of $n$+1 except when $n$+1 is a prime or $n$=3. The situation for $\underline{q}_n!$ is more complex: from [18] we know that

$$\underline{q}_{n+1} = \underline{q}_m \underline{(q^m)}_r$$

where $n+1=mr$. So

$$\underline{q}_{n+1} \mid \underline{q}_n! = \prod_{i=0}^{n-1} \underline{q}_{n-1}.$$

Instead of investigating this further, we define $\beta_n$ by

$$t / f(t) = \sum_{n=0}^{\infty} \beta_n t^n / \underline{q}_n!,$$

And we know from (5.4) and (5.5) that

$$t/f(t) \equiv \sum_{n=0}^{\infty} \frac{e_{n+1}}{\underline{q}_{n+1}} \sum_{r=1}^{\infty} (a_{n+1})^{r-1} \frac{t^{rn}}{\underline{q}_{rn}!} \pmod{\underline{q}_{n+1}}$$

$$\equiv \sum_{m=1}^{\infty} \frac{e_m}{\underline{q}_m} \sum_{r=1}^{\infty} (qa_m)^{r-1} \frac{t^{r(m-1)}}{\underline{q}_{rn}!} \pmod{\underline{q}_m}.$$

Thus, from the definition of $\beta_n$, we get for $z>0$, $m>2$,

$$\underline{q}_m \beta_n \equiv \begin{cases} e_m (qa_m)^{(n/(m-1)-1)} \pmod{\underline{q}_m} & (m-1 \mid n), \\ 0 \pmod{\underline{q}_m} & (m-1 \nmid n). \end{cases} \tag{5.8}$$

We next show that

$$a_m + e_m \equiv 0 \pmod{\underline{q}_m}.$$

From (4.9) and (5.5)

$$0 = \sum_{n=1}^{m} e_n D_{0q}^m f^n(t)/\underline{q}_n!,$$

which becomes

This simplifies to

$$a_m + e_m \underline{q}_{m-1}!/\underline{q}_{m-1}! \equiv 0 \pmod{\underline{q}_m}$$

from (4.10) and because

$$D_{0q}^m f(t) \equiv a_m \pmod{\underline{q}_m}.$$

Thus,

$$a_m + e_m \equiv 0 \pmod{\underline{q}_m},$$

and so, (5.8) becomes

$$\underline{q}_m \beta_n \equiv \begin{cases} -q^{(n/(m-1)-1)} a_m^{n/(m-1)} \pmod{\underline{q}_m} & (m-1 \mid n), \\ 0 & (m-1 \nmid n), \end{cases}$$

for $m>2$, $q>0$. For

$$f(t) = E_q(t) - 1 \qquad \text{(see (2.1) and (2.2)}$$

$$= \sum_{n=1}^{\infty} t^n / \underline{q}_n!$$

$a_n = 1$ and $\beta_n = B_{n,q}$. Thus,

$$\underline{q}_m B_{n,q} \equiv \begin{cases} -q^{(n/(m-1)-1)} \pmod{\underline{q}_m} & (m-1 \mid n), \\ 0 & (m-1 \nmid n). \end{cases} \tag{5.9}$$

for $m>2$, $q>0$.

When $m=p$, a prime, and $q=1$, this reduces to (2.3). (5.9) is not a necessary and sufficient condition for Fermatian Bernoulli numbers though, because no conditions were imposed on the allowable values of $\underline{q}_m$. (5.9) is an analogue of the Staudt-Clausen Theorem and it exists for the Fermatian Bernoulli numbers $B_{n,q}$.

## 6. CONCLUSION

The foregoing does not exhaust possibilities for generalizing the Bernoulli numbers and the Staudt-Clausen Theorem. Vandiver [20], for instance, defined Bernoulli numbers of the first order by the umbral equality

$$b_n(m,k) = (mb+k)^n : B_n = b_n(0,k), \tag{6.1}$$

so that Vandiver's form of the Staudt-Clausen Theorem was that for *n* even,

$$b_n(m,k) = A_n - \sum_{i=1} \frac{1}{p_,}, \tag{6.2}$$

where the *p*s are distinct primes, relatively prime to non-zero *m* and such that

$$n \equiv 0 \,(\mathrm{mod}\; p_i - 1),$$

and $A_n$ is some integer. Sharma [19] and Carlitz [1,2,8,9,10,11] have also studied analogues of the Staudt-Clausen type. It is of interest to note that Carlitz speculated about the existence of a theorem of the Staudt-Clausen type for Bernoulli numbers of order *k* defined by

$$(t/(e^t - 1))^k = \sum_{n=0}^{\infty} B_n^{(k)} t^n / n!, \tag{6.3}$$

and he showed that for $k = \underline{p}_r$ and

$$n = p^{r-1}\big(s(p-1)+1\big)-1$$

a form exists, namely,

$$p^r B_n^{\underline{p}_r} \equiv (-1)^r \;\; (\mathrm{mod}\, p).$$

Furthermore,

$$B_{p+2}^{p+1} \equiv 0 \;(\mathrm{mod}\, \underline{p}_4).$$

Another possibility for further research is to study the reducibility of the generalized Bernoulli polynomials [6]. Carlitz has used the Staudt-Clausen Theorem and Lagrange's Interpolation Formula to show that the polynomial in $x, pB_{p-1}(x)/x$ is an Eisenstein polynomial, and hence irreducible. This is also suggests the formal consideration of the *p*th Fermatian of index $x$, $\underline{x}_p$, as the irreducible cyclotomic polynomial, $\phi_p(x)$:

$$\underline{x}_p = \phi_p(x) = 1 + x + x^2 + \ldots + x^{p-1},$$

which satisfies the hypotheses of the Eisenstein criterion.

## REFERENCES


1. L. Carlitz. "An Analogue of the von Staudt-Clausen Theorem." *Duke Mathematical Journal*. **3** (1937): 503-517.

2. L. Carlitz. "An Analogue of the Staudt-Clausen Theorem." *Duke Mathematical Journal*. **7** (1941): 62-67.
3. L. Carlitz. "The Coefficients of the Reciprocal of a Series." *Duke Mathematical Journal*. **8** (1941): 689-700.
4. L. Carlitz. "q-Bernoulli Numbers and Polynomials." *Duke Mathematical Journal*. **15** (1948): 987-1000.
5. L. Carlitz. "Some Properties of Hurwitz Series." *Duke Mathematical Journal*. **16** (1949): 285-295.
6. L. Carlitz. "A Note on Irreducibility of the Bernoulli and Euler Polynomials." *Duke Mathematical Journal*. **19** (1952): 475-481.
7. L. Carlitz. "The Schur Derivative of a Polynomial." *Proceedings of the Glasgow Mathematical Association*. **1** (1953): 159-163.
8. L. Carlitz. "A Degenerate Staudt-Clausen Theorem." *Archiv der Mathematik*. **7** (1956): 28-33.
9. L. Carlitz. "A Note on the Staudt-Clausen Theorem." *American Mathematical Monthly*. **64** (1957): 186-188.
10. L. Carlitz. "Criteria for Kummer's Congruences." *Acta Arithmetica*. **6** (1960): 375-390.
11. L. Carlitz. "The Staudt-Clausen Theorem." *Mathematics Magazine*. **35** (1961): 131-146.
12. L. Carlitz. "Some Arithmetic Sums Associated with the Greatest Integer Function." *Archiv der Mathematik*. **12** (1961): 34-42.
13. G. Fontené. "Generalization d'une formule connue." *Nouv. Ann. Math.* **15** (1915): 112.
14. A.F. Horadam & A.G. Shannon. "Ward's Staudt-Clausen Problem." *Mathematica Scandinavica*. **39** (1976): 239-250.
15. F. H. Jackson. "q-difference Equations." *American Journal of Mathematics*. **32** (1910): 305-314.
16. A. G. Shannon. "Arbitrary Order Circular Functions: An Extension of Results of Glaisher and Lucas." *Journal of Natural Science & Mathematics*. **19** (1979): 71-76.
17. A. G. Shannon. "Some Fermatian Special Functions." *Note on Number Theory & Discrete Mathematics*. **9** (2003): 73-82.
18. A. G. Shannon. "Some Properties of Fermatian Numbers." *Note on Number Theory & Discrete Mathematics*: **10** (2004), 2: 25-33.
19. A. Sharma. "*q*-Bernoulli and Euler Numbers of Higher Order." .*Duke Mathematical Journal*. **25** (1958): 343-353.
20. H. S. Vandiver. "Simple Explicit Expressions for Generalized Bernoulli Numbers." *Duke Mathematical Journal*. **8** (1941): 575-584.
21. M. Ward. "A Calculus of Sequences." *American Journal of Mathematics*. **58** (1936): 255-266.


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