A FERMATIAN STAUDT-CLAUSEN THEOREM

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ABSTRACT
This paper looks at the Staudt-Clausen theorem within the framework of various
generalization of the Bernoulli numbers. The historical background to the prob-
lem is reviewed, and a solution to a problem of Morgan Ward is put forward. Generalized Hurwitz series are utilised in the development of the results.

1. INTRODUCTION
Morgan Ward [21] once posed the problem whether a suitable definition for generalized
Bernoulli numbers could be framed so that a generalized Staudt-Clausen Theorem might
exist for them within the framework of the Fontené-Jackson calculus [6,8].

Carlitz [4] outlined a partial generalization of the Staudt-Clausen theorem with the Fon-
téné-Jackson operators. The purpose of this paper is to show that Ward’s problem can be
solved with the adaptation of a method used by Carlitz for the ordinary Staudt-Clausen

2. DEFINITIONS
To define these generalized Bernoulli numbers, $B_{n, q}$, we use the $n$th reduced Fermatians
of index $q$ as defined in Shannon [17], namely:

$$\frac{t}{E_q(t) - 1} = \sum_{n=0}^{\infty} B_{n, q} t^n / q_n !, \quad (2.1)$$

where

$$E_q(t) = \sum_{n=0}^{\infty} t^n / q_n !, \quad (2.2)$$

and

$$q_n = \begin{cases} - q^n q^{-n} & (n < 0) \\ 1 + q + q^2 + ... + q^{n-1} & (n > 0) \\ 1 & (n = 0) \end{cases}$$

so that

$$q_n! = q_n q_{n-1}!,$$

and

$$1_n = 1, 1_n! = n!.$$

Thus,
the ordinary Bernoulli numbers, $B_n$, for which a form of the Staudt-Clausen theorem can be stated as follows:

\[
p_{B_n} \equiv \begin{cases} 
  -1 \pmod{p} & (p - 1 \mid n) \\
  0 \pmod{p} & (p - 1 \nmid n)
\end{cases}
\] (2.3)

where $p$ is an odd prime (and hence $n$ is even). Our analogue of (2.3) for the Fermatian Bernoulli numbers is in (5.9).

### 3. GENERALIZED DIFFERENTIAL OPERATORS

Carlitz also studied generalized versions of differential operators in the form of Chak derivatives defined by

\[
D_q (fx) = \frac{f(qx) - f(x)}{qx - x}.
\] (3.1)

He has also investigated properties associated with the Schur derivative

\[
\Delta a_m = (a_{m+1} - a_m) / p^{m+1},
\] (3.2)

where $\{a_m\}$ is a sequence and $p$ is a prime number [7]. In the same spirit then it is convenient to define formally the Fermatian differential operator

\[
D_{x,q} = q^n x^{n-1}.
\] (3.3)

It follows that

\[
D_{x,q} x^n = nx^{n-1}.
\]

Some properties of the Fermatian differential operator follow if we define

\[
D_{q,s} = (1 - q^n) x^{n-1},
\] (3.4)

and

\[
D_{q,s} q = 0,
\]

so that

\[
(1 - q)D_{x,q} x^n = (1 - q^n)x^{n-1} = D_{q,s} x^n.
\] (3.5)

Then

\[
D_{x,q} ax^n = aD_{x,q} x^n,
\]

where $a$ is a constant, and for $f(y)$, a function of $y$,

\[
D_{x,q} f(y) = D_{y,q} f(y)D_{x,q} y,
\]
which reduces to the ordinary ‘function of a function rule’ when \( q \) is unity.

\[
D_y f(y) = D_x f(y) D_y y
\]

Other properties include

\[
D_q y^n = q^n y^{n-1} D_q y
\]

and

\[
D_q \left( x^n + y^n \right) = D_q x^n + D_q y^n,
\]

which is analogous to Leibnitz’ Theorem for the \( n \)th derivative of a product of two functions:

\[
D^n u v = \sum_{r=0}^{n} \binom{n}{r} D^r u D^{n-r} v,
\]

in which

\[
\binom{n}{r} = \frac{q^n}{q^n r! (q^n - r)!}.
\]

The proof of (3.6) follows readily by induction on \( n \). We also define the operator \( I_{sq} \) formally by

\[
I_{sq} D_q f(x) = f(x) \quad (3.7)
\]

and

\[
D_q f(x) = I^{-1}_{sq} f(x) \quad (3.8)
\]

Thus, for \( n \neq -1 \),

\[
I_{sq} = \frac{1 - q}{1 - q^{-n+1}} x^{n+1} + C
\]

\[
= \frac{x^{n+1}}{q^{n+1}} + C,
\]

in which \( C \) is a constant determined by the initial conditions, and, for \( n = -1 \), we have

\[
I_{sq}^{-1} = L_q (x) + C
\]

where

\[
L_q (1 + x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{r+1}}{q^r q^{r+1}}.
\]

is an analogue of the logarithmic function.
From (3.5) we have that
\[ I_{-1} f(x) = (1 - q)^{-1} D_{q} f(x), \]
and so we can introduce \( I_{q} \)
\[ I_{q}^{-1} f(x) = (1 - q)^{-1} I_{q}^{-1} f(x). \]
This means that
\[ I_{q} f(x) = D_{q}^{-1} f(x). \]
and
\[ I_{-q} f(x) = (1 - q) I_{q} f(x). \]
When \( q = 1 \),
\[ I_{x} x^{n} = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1. \]
so that \( I_{x} x^{n} \) and \( \int x^{n} dx \) differ by a constant only, which can be made zero with suitable limits; that is, the \( I \) operation is a generalization of integration. One can also define generalizations of the circular and hyperbolic functions in a somewhat similar manner [16].

4. GENERALIZED HURWITZ SERIES

We shall call series of the form
\[ \sum_{n=0}^{\infty} \frac{a_{n} t^{n}}{q_{n} !}, \]
where the \( a_{n} \) are arbitrary integers, a generalized Hurwitz series (GH-series). When \( q \) is unity we get an ordinary Hurwitz series [5]. If we consider another GH-series
\[ \sum_{n=0}^{\infty} \frac{b_{n} t^{n}}{q_{n} !}, \]
then the product of (4.1) and (4.2)
\[ \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} a_{r} b_{n-r} \right) t^{n} / q_{n} !, \]
is also a GH-series.

The Fontené-Jackson type derivatives and integrals of GH-series are also GH-series, namely
\[ D_{q} \sum_{n=0}^{\infty} \frac{a_{n} t^{n}}{q_{n} !} = \sum_{n=0}^{\infty} a_{n+1} t^{n} / q_{n} !, \]
and
\[
I_{ sq} \left[ \sum_{n=0}^{\infty} \frac{a_n x^n}{q_n} \right]_0 = \sum_{n=1}^{\infty} \frac{a_{n-1} t^n}{q_n} !.
\]

For a series without constant term such as
\[
H_1(t) = \sum_{n=1}^{\infty} \frac{a_n t^n}{q_n}!,
\]

it follows from
\[
H_1^k(t) / q_k ! = I_{ sq} D_{ sq} H_1^k(x) / q_k ! \bigg|_{0}^t = I_{ sq} H_1^{k-1}(x) D_{ sq} H_1(x) / q_{k-1} ! \bigg|_{0}^t
\]

that \( H_1^k(t) / q_k ! \) is a GH-series for all \( k \geq 1 \). This result can also be stated in the form
\[
H_1^k(t) \equiv 0 (\text{mod } q_k !)
\]

where by the statement
\[
\sum_{n=0}^{\infty} \frac{a_n t^n}{q_n} ! \equiv \sum_{n=0}^{\infty} \frac{b_n t^n}{q_n} ! \text{ (mod } q_m \)
\]
is meant that the system of congruences
\[
a_n \equiv b_n (\text{mod } q_m), \quad (n = 0,1,2,\ldots)
\]
is satisfied. This is equivalent to the assertion
\[
\sum_{n=0}^{\infty} \frac{a_n t^n}{q_n} ! = \sum_{n=0}^{\infty} \frac{b_n t^n}{q_n} ! + q_m H(t)
\]
where \( H(t) \) is some GH-series.

We now consider the class of GH-series
\[
f(t) = \sum_{n=1}^{\infty} \frac{a_n t^n}{q_n} !, \quad (a_1 = 1)
\]

and
\[
D_{ sq} = \sum_{n=0}^{\infty} A_n f^n(t), \quad (A_0 = 1)
\]
where the \( A_n \) are integers. It follows from (4.6) and (4.8) that
\[
D_{ sq} f(t) \equiv \sum_{n=0}^{m-1} f^n(t) \text{ (mod } q_m \)
\]

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since \( q_m \mid q_m^! A_k f^k(t) \), \( k \geq m \). It follows from (4.7) that \( f(0)=0 \), and so

\[
D_{0q} f(t) = 1
\]

where \( D_{0q} f(t) \) denotes the \( r \)th generalized Fontené-Jackson derivative of \( f(t) \) evaluated at \( t=0 \). A result we shall use is

\[
D_{0q}^m f^k(t) = 0 \pmod{q_m^\cdot} \quad (k>1). \tag{4.9}
\]

This arises because

\[
D_{0q}^m f^{k+1}(t) = D_{0q}^m f^k(t) f(t) = f(0)D_{0q}^m f^k(t) + f^k(0)D_{0q}^m f(t) \pmod{q_m^\cdot}.
\]

Another result to be used later is

\[
D_{0q}^m f^m(t) / q_m^\cdot = q_{m-1}^! \pmod{q_m^\cdot}. \tag{4.10}
\]

**Proof:**

\[
D_{0q}^m f^m(t) / q_m^\cdot = D_{0q}^{m-1} \left(D_{0q} f^m(t) / q_m^\cdot\right)
= D_{0q}^{m-1} \left(f^{m-1}(t) D_{0q} f(t)\right)
\equiv D_{0q}^{m-1} f^{m-1}(t) \pmod{q_m^\cdot}
\equiv D_{0q}^{m-2} \left(q_{m-1} D_{0q}^{m-2} f^{m-2}(t) / q_m^\cdot\right) \pmod{q_m^\cdot}
\equiv q_{m-1} D_{0q}^{m-2} f^{m-2}(t) \pmod{q_m^\cdot}
\equiv q_{m-1}^! \pmod{q_m^\cdot}.
\]

\[\square\]

5. FERMATIAN STAUDT-CLAUSEN THEOREM

If we put

\[
f^{m-1}(t) = \sum_{n=m-1}^{n} a_n^* t^n / q_m^! \tag{5.1}
\]

then, since \( q_m^\cdot \mid q_m^! f^m(t) \) from (4.5),

\[
f^{m-1}(t) f(t) \equiv 0 \pmod{q_m^\cdot}, \]

and we get

\[
\sum_{r=0}^{n} \left[ n \atop r \right] a_n^* a_{n-r} \equiv 0 \pmod{q_m^\cdot}, \quad n \geq q_m^\cdot. \tag{5.2}
\]

Note that (4.7) and (5.1) imply that
\[ a'_{m-1} = q_{m-1}^{-1}. \]

From the definition of \( a_n \) and \( a'_n \), for \( n=m=1 \), Congruence (5.2) reduces to
\[ q_{m+1} a_{m} a'_m \equiv 0 \pmod{q_m} \]
so that
\[ a'_m \equiv 0 \pmod{q_m} \]
(from [18]).

For \( n=m+2 \), (5.2) becomes
\[ q_{m+2} a_1 a'_{m+1} + \left[ m+2 \atop 2 \right] a_2 a'_m \equiv 0 \pmod{q_m} \]
which simplifies to
\[ q_{m+2} a_1 a'_{m+1} \equiv 0 \pmod{q_m} \]
so that
\[ a'_{m+1} \equiv 0 \pmod{q_m} \]
since
\[ q_{m+2} \equiv 1 + q \pmod{q_m} \]
(see [18])
and
\[ 1 \equiv (1 + q, q_m) \]
\((q > 0, m > 2)\).

Continuing in this way, we get
\[ a'_{m} \equiv a'_{m+1} \equiv \ldots \equiv a'_{2m-3} \equiv 0 \pmod{q_m}. \]

For \( n=2m-1 \), we get
\[ q_{2m-1} a_1 a'_{2m-2} + \ldots + \left[ 2m-1 \atop m \right] a_m a'_{m-1} \equiv 0 \pmod{q_m} \]
which gives
\[ q_{2m-1} a'_{2m-2} + a_{m} a'_m \equiv 0 \pmod{q_m} \]
(see [18])
so that
\[ a'_{2m-2} \equiv q a_m a'_{m-1} \equiv q a_m q_{m-1}^{-1} \pmod{q_m}. \]

From (5.1) we get
\[ D_{q} q^{n-1} f^{m-1}(t) = \sum_{n=m-1}^{\infty} a'_n t^{n-m+1} / q_{n-m+1}^{-1} \] 
\[ = a'_{m-1} + a'_{2m-2} t^{m-1} / q_{m-1}^{-1} + \ldots \]
\[ = q_{m-1}^{-1} + q a_m \left( a'_{m-1} t^{m-1} / q_{m-1}^{-1} + \ldots \right) \]
\[ = q_{m-1}^{-1} + q a_m f^{m-1}(t) \pmod{q_m}. \]

A solution of (5.3) is
\[ f^{m-1}(t) \equiv q_{m-1}^{-1} \sum_{n=1}^{\infty} (qa_m)^{n-1} t^{n(m-1)} / q_{n(m-1)}^{-1} \pmod{q_m}. \]

(5.4)
This can be verified as follows
\[
D_{m^{n-1}} f^{m-1}(t) = q_{m^{-1}} \sum_{n=1}^{\infty} \left( q a_m \right)^{n-1} \frac{t^{(n-1)(m-1)}}{q_{(n-1)(m-1)!}}
\]
\[
\equiv q_{m^{-1}} \sum_{n=0}^{\infty} \left( q a_m \right)^n \frac{t^{n(m-1)}}{q_{n(m-1)!}}
\]
\[
\equiv q_{m^{-1}}! + q a_m f^{m-1}(t) \pmod{q_m}.
\]

Now let
\[
\lambda(t) = \sum_{n=1}^{\infty} e_n f^n(t) / q_n!.
\]

\((e_1 = 1)\)

denote the inverse of \(f(t)\), so that
\[
t = \sum_{n=1}^{\infty} e_n f^n(t) / q_n!.
\]

(5.5)

Differentiating (5.5) we get
\[
1 = \sum_{n=0}^{\infty} e_{n+1} f^n(t) / q_n! D_{q} f(t).
\]

Comparison with (4.8) yields
\[
e_{n+1} \equiv 0 \pmod{q_n!}.
\]

(5.6)

(5.5) can be re-written as
\[
t / f(t) = \sum_{n=0}^{\infty} e_{n+1} f^n(t) / q_{n+1}!.
\]

(5.7)

It follows from (4.5) and (5.6) that \(e_{n+1} f^n(t) / q_{n+1}!\) is a GH-series and the coefficients of \(f^n(t)\) are multiples of \(q_{n+1}!\). At this stage with ordinary integers one can proceed by using the fact that \(n!\) is a multiple of \(n+1\) except when \(n+1\) is a prime or \(n=3\). The situation for \(q_n!\) is more complex: from [18] we know that
\[
q_{n+1} = q_m \left( q_n \right)_r.
\]

where \(n+1=mr\). So
\[
q_{n+1}! = \prod_{j=0}^{m-1} q_{q_j!}.
\]

Instead of investigating this further, we define \(\beta_n\) by
\[
t / f(t) = \sum_{n=0}^{\infty} \beta_n t^n / q_n!.
\]

And we know from (5.4) and (5.5) that
\[ t / f(t) = \sum_{n=0}^{\infty} e_{n+1} \sum_{r=1}^{\infty} \left( a_{n+1} \right)^{r-1} \frac{t^m}{q_m^r} \text{ (mod } \frac{q_{n+1}}{q_m} \text{)} \]

\[ = \sum_{m=1}^{\infty} e_m \sum_{r=1}^{\infty} \left( q a_m \right)^{r-1} \frac{t^m}{q_m^r} \text{ (mod } \frac{q_m}{q_m} \text{)} . \]

Thus, from the definition of \( \beta_n \), we get for \( z>0, m>2, \)

\[ q_m \beta_n = \begin{cases} e_m \left( q a_m \right)^{n(m-1)-1} \text{ (mod } \frac{q_m}{q_m} \text{)} & (m-1 \mid n) , \\ 0 \text{ (mod } \frac{q_m}{q_m} \text{)} & (m-1 \not\mid n) \end{cases} \]  

(5.8)

We next show that

\[ a_m + e_m \equiv 0 \text{ (mod } \frac{q_m}{q_m} \text{).} \]

From (4.9) and (5.5)

\[ 0 = \sum_{n=1}^{m} e_n D_{0q}^m f''(t) / q_n^1 , \]

which becomes

This simplifies to

\[ a_m + e_m q_m^{-1} / q_m^{-1} \equiv 0 \text{ (mod } q_m \text{)} \]

from (4.10) and because

\[ D_{0q}^m f(t) \equiv a_m \text{ (mod } q_m \text{).} \]

Thus,

\[ a_m + e_m \equiv 0 \text{ (mod } q_m \text{)}, \]

and so, (5.8) becomes

\[ q_m \beta_n = \begin{cases} -q^{n(m-1)-1} a_m \text{ (mod } q_m \text{)} & (m-1 \mid n) , \\ 0 \text{ (mod } q_m \text{)} & (m-1 \not\mid n) \end{cases} \]

for \( m>2, q>0 \). For

\[ f(t) = E_q(t) - 1 \]

(see (2.1) and (2.2))

\[ = \sum_{n=1}^{\infty} t^n / q_n^1 \]

\[ a_n = 1 \text{ and } \beta_n = B_{n,q} . \]  Thus,

\[ q_m B_{n,q} = \begin{cases} -q^{n(m-1)-1} \text{ (mod } q_m \text{)} & (m-1 \mid n) , \\ 0 \text{ (mod } q_m \text{)} & (m-1 \not\mid n) \end{cases} \]  

(5.9)

for \( m>2, q>0 \).

When \( m=p \), a prime, and \( q=1 \), this reduces to (2.3). (5.9) is not a necessary and sufficient condition for Fermatian Bernoulli numbers though, because no conditions were imposed on the allowable values of \( q_m \). (5.9) is an analogue of the Staudt-Clausen Theorem and it exists for the Fermatian Bernoulli numbers \( B_{n,q} \).
6. CONCLUSION

The foregoing does not exhaust possibilities for generalizing the Bernoulli numbers and the Staudt-Clausen Theorem. Vandiver [20], for instance, defined Bernoulli numbers of the first order by the umbral equality

\[ b_n(m, k) = (mb + k)^n : B_n = b_n(0, k), \]

so that Vandiver’s form of the Staudt-Clausen Theorem was that for \( n \) even,

\[ b_n(m, k) = A_n - \sum_{i=1}^{\infty} \frac{1}{p_i}, \]

where the \( p_i \)s are distinct primes, relatively prime to non-zero \( m \) and such that

\[ n \equiv 0 \pmod{p_i - 1}, \]

and \( A_n \) is some integer. Sharma [19] and Carlitz [1,2,8,9,10,11] have also studied analogues of the Staudt-Clausen type. It is of interest to note that Carlitz speculated about the existence of a theorem of the Staudt-Clausen type for Bernoulli numbers of order \( k \) defined by

\[ (t/(e^t - 1))^k = \sum_{n=0}^{\infty} \frac{B_n^{(k)} t^n}{n!}, \]

and he showed that for \( k = p^r \) and

\[ n = p^{r-1}(s(p-1)+1)-1 \]

a form exists, namely,

\[ p^r B_{-1}^{\frac{p}{s}} \equiv (-1)^r \pmod{p}. \]

Furthermore,

\[ B_{p+2}^{p+1} \equiv 0 \pmod{p_4}. \]

Another possibility for further research is to study the reducibility of the generalized Bernoulli polynomials [6]. Carlitz has used the Staudt-Clausen Theorem and Lagrange’s Interpolation Formula to show that the polynomial in \( x, pB_{p-1} (x)/x \) is an Eisenstein polynomial, and hence irreducible. This is also suggests the formal consideration of the \( p \)th Fermatian of index \( x, x_p \), as the irreducible cyclotomic polynomial, \( \phi_p (x) \):

\[ x_p = \phi_p (x) = 1 + x + x^2 + ... + x^{p-1}, \]

which satisfies the hypotheses of the Eisenstein criterion.

REFERENCES


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