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USING INTEGER STRUCTURE TO CALCULATE THE NUMBER OF PRIMES IN A GIVEN INTERVAL

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Abstract

With the exception of 2 and 3, primes are only found in two classes of the modular ring Z_6. The rows of the tabular display of this ring which contain composites in these two classes are given by $R = R_0 + pt$, $t = 0,1,2,3,...,$ in which R_0 is a function of p that is class specific. The number of composites, n_c, in the two classes can be calculated from:

$$n_c \in \bar{2}_6 = \left\{ \sum_{p}^{p_L} \left[\left(\left(N - p^2 \right) / 2p \right) + a \right] / 3 \right\} - \left(Q_1 + Q_2 \right)$$

(a=1 or 2 depending on the class of p),

$$n_C \in \bar{4}_6 = \left\{ \sum_{p}^{p_L} \left[\left(\left(N - p^2 \right) / 6p \right) + 1 \right] - \left(Q_3 \right) \right\}$$

where p_L is the prime less than and closest to $\sqrt{N}$. The Q_i terms are quantities which arise from the characteristics of the factors of the composites. Subtraction of n_c from the total integers in each class yields the number of primes for that class, and hence the total number of primes in the interval.

1. Introduction

We have previously investigated the importance of integer structure in the analysis of such quantities as primes, squares, triples and Fibonacci numbers [1-4]. Here we demonstrate how an analysis of the composite-building by primes permits accurate calculation of the number of primes in the interval [1,*N*]. The modular ring Z_6 is used as the most appropriate integer structure for this purpose.

Integers in Z_6 are represented by $(6r_i + (i-3))$ where $\bar{i}$ is the class. The three classes $\bar{2}_6, \bar{4}_6$ and $\bar{6}_6$ contain odd integers. Class $\bar{2}_6$ contains no even powers, while all

$N \in \overline{3}_6, \overline{6}_6$ are such that $3|N$. Even integers fall in Classes $\overline{1}_6, \overline{3}_6, \overline{5}_6$, the last having no even powers.

Composite odd integers in Classes $\overline{2}_6$ and $\overline{4}_6$ (which contain all the primes except 2 and 3) fall in rows of the array of Classes, R [1] (Table 1):

$$R = R_0 + pt, \ \ t = 0,1,2,3,... \tag{1.1}$$

Class of *M*	**Class of *p***	R_0
$\overline{4}_6$	$\overline{2}_6$ & $\overline{4}_6$	$(p^2-1)/6$
$\overline{2}_6$	$\overline{2}_6$ $\overline{4}_6$	$\left(\frac{1}{2}(p^2+1)+p\right)/3$ $\left(\frac{1}{2}(p^2+1)+2p\right)/3$

Table 1: The R_0 functions

2. Primes in Class $\overline{4}_6$

Calculation of the total number, n_c, of composite integers in this class is given by

$$n_C = \left\{ \sum_{p}^{p_L} \left[\left(\left(N - p^2 \right) / 6p \right) + 1 \right] - Q \right\}, \tag{2.1}$$

in which N is the last of the integers in the chosen range from 1 to N, and P_L is the prime which directly precedes $\sqrt{N}$. Equation (2.1) comes from the composite-number equation [1]:

$$M = 6\{((p^2-1)/6) + pt\} + 1. \tag{2.2}$$

The braced term is the "forbidden" row which contains this composite. The Q term in Equation (2.1) arises because composites can have more than one factor so that the sum overestimates these integers. For example, if the composite has factors 5 and 7, then it will appear in the p=5 and p=7 lists. However, such duplications can be readily identified and corrected. The estimate of Q proceeds as follows.

Suppose that the series for p_1 and p_2 both contain M, then

$$M = p_1^2 + 6p_1t_1$$
$$= p_2^2 + 6p_2t_2,$$

so that

$$t_1 = \left(p_2^2 - p_1^2 + 6p_2t_2\right)/6p_1, \ \ p_1 < p_2, \tag{2.3}$$

with maximum t_2 given by

$$\hat{t}_2 = \left(N - p_2^2\right)/6p_2, \text{ (Table 2)}.$$

when for the t_2 range $[0, \hat{t}_2]$, integer solutions are found for t_1, and this gives the components of Q for p_2. However, as p_1 varies with p_2, repeats of t_2 must be discarded if Q is to be estimated accurately. For instance, let *N*=1105. Then $\sqrt{N} = 33.2154 \approx 33$, so that $p_L = 31$. Thus only the primes {5,7,11,13,17,19,23,29,31} are needed to form all the integers in Classes $\overline{2}_6$ and $\overline{4}_6$ up to 1105. The trick here is that only the smallest prime factor of the composite is needed for identification of the row of the composite in Z_6 (Equation (1.1)) and hence its value from 6*r*±1, the Class being known ($\overline{2}_6$ or $\overline{4}_6$). Obviously, the smallest prime will be less than $\sqrt{N}$, no matter how large *N* is. When the composite has two or more factors less than or equal to p_L, then *Q* values need to be considered. From Equation (2.1), the total number of composites plus *Q*, $(n_c + Q) \in \overline{4}_6$, is given by

$$n_c + Q = \sum_{p=5}^{31} \{((1105 - p^2)/6p) + 1\} = 115, \text{ (Table 3).} \qquad (2.4)$$

p	5	7	11	13	17	19	23	29	31
$\hat{t}$	-	25	14	12	8	6	4	1	0

Table 2: Maximum *t* for *N*=1105

p	5	7	11	13	17	19	23	29	31	
Sum	37	26	15	13	9	7	5	2	1	=115

Table 3: Breakdown of Equation (2.4)

Note that each component of the sum is rounded down to the integer (the floor function). In Table 4 we list the t_2 values for each *p* set that has the same *M* as another of the *p* sets. The *t* values cannot be repeated for a given set and these are discarded, otherwise *Q* will be overestimated.

$p_1 \downarrow$ $p_2 \rightarrow$	7	11	13	17	19	23	29	31
5	3,8,13,18,23	4,9,14	2,7,12	3,8	1,6	2	1	-
7		~~4~~,11	6	~~3~~	5	~~2~~	~~1~~	
11			~~7~~	-	~~6~~	-	~~1~~	
13				~~8~~	-	-	-	
17								
19								
23								
29								

Table 4: Acceptable values of t_2 $(p_1 < p_2)$ in example

As can be seen from Table 4 there are twenty acceptable values of t_2, so that *Q*=20. Since 1105=(6x184+1), there will be 185 rows (including the zeroth row) and hence 185 integers in Class $\overline{4}_6$ (1-1105 range). The total composites will be 115 (from the sum) plus 1 (1 being the first integer in the Class) minus 20, which gives 96 composites. Thus there must be 89 primes in this range (185-96) for this Class. This is the correct value.

It is of interest to note that the t_2 values are given by

$$t_2 = \breve{t}_2 + np_1,\ n = 0,1,2,3,...,\ \text{until } \hat{t}_2 \text{ is reached,}$$

where $\breve{t}_2$ is the minimum value. Thus (n+1) gives the number of valid solutions (rounded to the integer); these are also given by $\{[\hat{t}_2 - \breve{t}_2]/p_1\}+1$. Of course, multiple t_2 values for a given p_1 must be taken into account, so that actual values of t_2 need to be scanned for multiplicity.

The value of Q involves limited numbers of p_1 because of the t_2 (limit) and duplication constraints (Table 4). When N=5005, only p_1=5,7,11 and 13 contribute to Q. For this value of N the sum =663 and Q=159, so that total composites in $\bar{4}_6$ (including 1, the first integer in $\bar{4}_6$) is 505. N falls in row 834 so that there are 835 integers in $\bar{4}_6$ in this range. Thus the number of primes in this Class is 835 – 505 =330.

3. Primes in Class $\bar{2}_6$

The analysis for this Class is more complex as can be seen in Table 1. There are two sets to analyse in turn.

Set A: Composites in $\bar{2}_6$ obtained from primes in $\bar{4}_6$

When the primes which form the composite come from Class $\bar{4}_6$ (7,13,19,31 in the example), then

$$M = p^2 + 2p(2+3t) \tag{3.1}$$

which comes from M=6R-1 and Table 1.

The total number of composites $n_c \in \bar{2}_6$ is given by

$$n_c = \left\{\sum_{p}^{p_L}\left[\left(\left(N-p^2\right)/2p\right)+1\right]/3\right\} - Q_{44} - Q_{42} \tag{3.2}$$

in which $p \in \bar{4}_6$.

The two Q terms arise because duplication of composites can occur not only within Set A but also between Sets A and B. When a common M occurs in Set A:

$$p_1^2 + 2p_1(2+3t_1) = p_2^2 + 2p_2(2+3t_2), \tag{3.3}$$

so that

$$t_1 = \left((p_2 - p_1)(p_2 + p_1 + 4) + 6p_2t_2\right)/6p_1 \tag{3.4}$$

with

$$\hat{t}_2 = \left(\left(\left(N - p^2\right)/2p\right) - 2\right)/3.$$

Table 5 lists the t_2 value fitting Equation (3.4).

When $t_2 \in B$, for which

$$M = p^2 + 2p(1+3t),$$

then

$$p_1^2 + 2p_1(2+3t_1) = p_2^2 + 2p_2(1+3t_2), \tag{3.5}$$

so that

$$t_1 = \left(p_2^2 - p_1^2 + 2(p_2 - 2p_1) + 6p_2t_2\right)/6p_1 \tag{3.6}$$

with

$$\hat{t}_2 = \left(\left(\left(N - p^2\right)/2p\right) - 1\right)/3..$$

Solutions are listed in Table 6. For Set A, the sum (Equation (3.2)) equals 44, while $Q_{42} + Q_{44} = 6$ so that $n_c = 38$ in this set.

Set B: Composites in $\bar{2}_6$ obtained from primes in $\bar{2}_6$

When the prime factors which form the composites come from Class $\bar{2}_6$ (that is, 5,11,17,23,29), then from Table 1 and the fact that

$$M = 6R - 1,$$

M is given by

$$M = p^2 + 2p(1+3t). \tag{3.7}$$

The total number of composites is

$$n_c = \left\{ \sum_{p}^{p_L} \left(\left(\left(N - p^2\right)/2p\right) + 2\right)/3 \right\} - Q_{22} - Q_{24}, \tag{3.8}$$

in which Q_{22} is obtained from solutions of

$$t_1 = \left((p_2 - p_1)(p_2 + p_1 + 2) + 6p_2t_2\right)/6p_1. \tag{3.9}$$

This simplifies because 6 is always a factor in the numerator, with maximum

$$\hat{t}_2 = \left(\left(\left(N - p^2\right)/2p\right) - 1\right)/3.$$

For Q_{24}, t_2 comes from the A set so that, in this case,

$$t_1 = \left(p_2^2 - p_1^2 + 2(2p_2 - p_1) + 6p_2t_2\right)/6p_1. \quad (3.10).$$

and

$$\hat{t}_2 = \left(\left(\left(N - p^2\right)/2p\right) - 2\right)/3..$$

Solutions are listed in Tables 7 and 8.

For the B set, the sums equal 65 and $Q_{22} + Q_{24} = 13$, so that $n_c = 52$. The last integer in $\bar{2}_6$ in the given range is 1103=6x184-1. This yields the total number of integers in this

Class as 184 (R=0 has M=-1). Since the A+B sums equal 109 and the Q terms sum to 19, the total number of composites in $\overline{2}_6$ will be 109-19=90. Thus the number of primes in $\overline{2}_6$ will be 184-90=94, which is correct.

p for $t_2 \rightarrow$ / p for $t_1 \downarrow$	13	19	31
7	3,10	2	–
13		–	–
19			–

Table 5: t_2 values for common M with $p_2 \in 4_6$; $Q_{44} = 3$

p for $t_2 \rightarrow$ / p for $t_1 \downarrow$	11	17	23	29
7	6,13	5	–	–
13		–	–	–
19			–	–
31				–

Table 6: t_2 values for common M with $p_2 \in \overline{2}_6$ (*cross matching*); $Q_{42} = 3$

p for $t_2 \rightarrow$ / p for $t_1 \downarrow$	11	17	23	29
5	2,7,12	1,6	0	–
11		6	–	–
17			–	–
23				–

Table 7: t_2 values for common M with $p_2 \in \overline{2}_6 ; Q_{22} = 6$

p for $t_2 \rightarrow$ / p for $t_1 \downarrow$	7	13	19	31
5	4,9,14,19, 24	3,8	2	0
11		10	–	–
17			–	–
23				
29				

Table 8: t_2 values for common M with $p_2 \in \overline{4}_6$ (*cross matching*); $Q_{24} = 7$

Total number of primes in the interval 1-1105

The prime p=2 falls in Class $\overline{5}_6$ and p=3 falls in Class $\overline{6}_6$, these being the only primes in these classes. Thus the total number of primes in [1,1105] is

$$n_p = 2 + 89 + 94 = 185.$$

This means that the primes average one per row in this interval, but

$$n_p \approx N / \ln N,$$

which yields 157. This falls 28 short, which compares with the total Q=39.

4. Final Comments

The foregoing analysis illustrates how the integer structure is highly Class specific, and this needs to be taken into account in the study of primes. For instance, the fact that more primes are of the form $4r_3 + 3$ compared with $4r_1 + 1$, or that primes in Class $\bar{2}_6$ are more plentiful than those in Class $\bar{4}_6$ (as shown above, 94 versus 89), may largely be explained in the latter case by noting that Class $\bar{2}_6$ has no even-powered odd integers. This means that Class $\bar{4}_6$ has to accommodate these and so there is less room for the primes. The same is true for the modular ring Z_4 in which the Class $\bar{3}_4$ has no even powers, so that, unless the integer regime under study has a dearth of such powers, the number of primes of the form $4r_3 + 3$ will exceed those of the form $4r_1 + 1$.

One of the great unsolved challenges in mathematics is the Riemann hypothesis. Since, as shown here and elsewhere [1], integer structure is the key to the distribution of the primes, and L-functions (which are types of zeta functions) [6] take this into account, then it would seem that analysis of L-functions would be the most appropriate way to pursue this challenge, particularly using the L-function related to primes of the form $6r\pm1$.

References

1. J.V. Leyendekkers & A.G. Shannon, The Analysis of Twin Primes within Z_6. *Notes on Number Theory & Discrete Mathematics*. **7(4)** 2001: 115-124.
2. J.V. Leyendekkers & A.G. Shannon, Powers as a Difference of Squares: The Effect on Triples. *Notes on Number Theory & Discrete Mathematics*. **8(3)** 2002: 95-106.
3. J.V. Leyendekkers & A.G. Shannon, Some Characteristics of Primes within Modular Rings. *Notes on Number Theory & Discrete Mathematics*. **9(3)** 2003:49-58.
4. J.V. Leyendekkers & A.G. Shannon, Analysis of Quadratic Diophantine Equations with Fibonacci-number Solutions. *International Journal of Mathematical Education in Science & Technology*. In press.
5. J.V. Leyendekkers & A.G. Shannon, Structure of Fibonacci Numbers within Modular Rings. *Notes on Number Theory & Discrete Mathematics*. In press.
6. Karl, Sabbagh, *Dr Riemann's Zeros.* London: Atlantic Books, 2002.

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