NNTDM 10 (2004), 3, 72-76

EULER'S PRIME EQUATION WITHIN THE COMPOSITE GRID OF THE MODULAR RING Z_6

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Abstract

The extended Euler-prime-generating equation is shown to be compatible with the composite grid of the modular ring Z_6. Invalid values of the variable x (those x values which yield composites) follow regular sequences within the grid and have associated couples within the modular ring which are linked to the prime values.

1. Introduction

We have recently shown [2] that Euler's prime generating equation

$$N = x^2 + x + p \tag{1.1}$$

where p=2,3,5,11,17,41, $0 \le x \le p-2$, can be extended when composite-producing x values are identified. Equation (1.1) does not seem to have any relationship with the equation

$$N = 6R \pm 1 \tag{1.2}$$

where

$$R = R' + pt,$$

with R' in Table 1.

Class of N	Class of p	R'
$\bar{4}_6$ $N = 6R+1$	$\bar{2}_6$ $\bar{4}_6$	$\frac{1}{6}\left(p^2-1\right)$
$\bar{2}_6$ $N = 6R-1$	$\bar{2}_6$ $\bar{4}_6$	$\frac{1}{3}\left(\frac{1}{2}\left(p^2+1\right)+p\right)$ $\frac{1}{3}\left(\frac{1}{2}\left(p^2+1\right)+2p\right)$

Table 1: Values for R'

Equation (1.2) is based on the modular ring Z_6 within which the integers N are given by $\left(6r_i + (i-3)\right)$, $\bar{i}$ representing the class. For example, $(6r_1 - 2) \in \bar{1}_6$ yields even integers within that Class. The primes are confined to $(6r_2 - 1) \in \bar{2}_6$ and $(6r_4 + 1) \in \bar{4}_6$. Equation

(1.2) allows rows which contain composites in either $\bar{2}_6$ or $\bar{4}_6$ to be identified. This implies that 'missed' rows in each Class must contain primes

In this paper, we show that composite-producing *x* values have corresponding (*p*,*t*) couples (Equation (1.2)) which yield the same composite. For a given *p* (smallest prime factor of *N*) it is shown that *x* follows simple sequences; this facilitates identification of composite-producing *x* values.

2. Euler's Equation in Z_6

<u>Case 1: x odd</u>

$x \in \{\bar{2}_6, \bar{4}_6, \bar{6}_6\}$. Integers $M \in \bar{6}_6$ are such that 3|*M*, and there are no even powers in $\bar{2}_6$. We note that when $p = 41, p \in \bar{2}_6$. Thus we have three possible solutions for *N*:

$$N = \bar{6}_6 + \bar{6}_6 + \bar{2}_6 = \bar{2}_6 \tag{2.1}$$

$$N = \bar{4}_6 + \bar{2}_6 + \bar{2}_6 = \bar{2}_6 \tag{2.2}$$

$$N = \bar{4}_6 + \bar{4}_6 + \bar{2}_6 = \bar{4}_6 . \tag{2.3}$$

Thus, when *x* is odd, $3 \nmid N$, and *N* falls most commonly in $\bar{2}_6$. Note that when $x \in \bar{2}_6, x^2 \in \bar{4}_6$, because $\bar{2}_6$ contains no even exponents.

<u>Case 2: *x* even</u>

$x \in \{\bar{1}_6, \bar{3}_6, \bar{5}_6\}$. Thus

$$N = \bar{1}_6 + \bar{1}_6 + \bar{2}_6 = \bar{4}_6 \tag{2.4}$$

$$N = \bar{1}_6 + \bar{5}_6 + \bar{2}_6 = \bar{2}_6 \tag{2.5}$$

$$N = \bar{3}_6 + \bar{3}_6 + \bar{2}_6 = \bar{2}_6 . \tag{2.6}$$

As with $\bar{2}_6, \bar{5}_6$ has no even powers. Again, $\bar{2}_6$ predominates and there are no 3|*N* results.

3. Values of *x* which Correspond to (*p*,*t*) Pairs in Equation (1.2)

Composite values of *N* obtained from Equation (1.1) have specific values of *x* [1]. We consider the smallest prime factor of *N*, *p* say, and with

$$N - 6(R' + pt) \pm 1 = x^2 + x + 41 \tag{3.1}$$

we can find the (*p*,*t*) couple which corresponds with *x*.

<u>Case 1: $N \in \bar{2}_6$</u>

(a) $p \in \bar{2}_6$. From Equation (3.1) and Table 1, we have

$$N = 6\left(\frac{1}{3}\left(\frac{1}{2}\left(p^2+1\right)+p\right)+pt\right)-1, \tag{3.2}$$

so that

$$t = \frac{1}{6p}\left(N - p^2 - 2p\right) \tag{3.3}$$

with

$$N = x^2 + x + 41.$$

(b) $p \in \bar{4}_6$. This gives

$$t = \frac{1}{6p}\left(N - p^2 - 4p\right). \tag{3.4}$$

For a given *p*, we have the fourth order difference equation

$$x_j = x_{j-4} + 3p \tag{3.5}$$

in both classes (Table 2).
Thus only the first five *x* values are needed for a given *p* in order to identify the higher *x* values.

p										
43	***x***	44	84	87	170	173	213	216	299	302
	t	0	20	22	105	109	169	174	340	347
61	***x***	65	117	126	239	248	300	309	422	431
	t	1	27	33	156	158	236	251	477	498
97	***x***	104	186	201	380	395	477	492		
	t	2	43	53	232	252	375	400		
131	***x***	140	252	383	402	533				
	t	3	59	165	184	340				

Table 2: Examples of (*p*,*t*) couples for a given *x*, $N \in \bar{2}_6$

Case 2: $N \in \bar{4}_6$

For this Class, the primes in $\bar{2}_6$ and $\bar{4}_6$ satisfy the same function (Table 1).
Thus,

$$N = 6\left(\frac{p^2-1}{6}+pt\right)+1 \tag{3.6}$$

so that

$$t = \frac{1}{6p}\left(N - p^2\right) \tag{3.7}$$

with

$$N = x^2 + x + 41.$$

For a given *p*, we have the second order difference equation

$$x_j = x_{j-2} + 3p\ . \tag{3.8}$$

Furthermore, there are fewer $\bar{4}_6$ composites which come from the *x* values (Section 2).

p								
47	***x***	49	91	190	232	331	373	472
	t	1	22	121	184	382	487	784
61	***x***	178	187	361	370	544		
	t	77	866	347	365	800		
71	***x***	76	136	289	349	502		
	t	2	32	185	275	581		
113	***x***	121	217	460	556			
	t	3	51	294	438			
173	***x***	184	334	703				
	t	4	79	448				

Table 3: Examples of (*p*,*t*) couples for a given *x*, $N \in \bar{4}_6$

4. Prime Sequences

The Euler equation fits into a specific pattern within the (*p*,*t*) grid of composites so that only some primes satisfy the (*p*,*t*) requirements of *x*. Even so, it is found that these primes follow regular series. For example, when $0 \le x \le 500$, there are two sets of these types of primes.

Set A. From Equation (1.1) with *N*=*p* (the fixed prime = 41),

$$x = \tfrac{1}{2}\left(-1 \pm \sqrt{1+4(p-41)}\right). \tag{4.1}$$

In this set, *x* is always integral for the given *p*, and *p* satisfies the sequence defined by the first order difference equation

$$p_j = p_{j-1} + 2j \tag{4.2}$$

with $p_0 = 41$; for example,

{*p*}={41,43,47,53,61,71,83,97,113,…}.

Set B. In this set *x* is non-integral in Equation (4.1) and *p* satisfies the sequence defined by the second order difference equation

$$p_j = 2p_{j-1} - p_{j-2} + 8 \tag{4.3}$$

with $p_0 = 163, p_1 = 167$; for example,

$$\{p\} = \{163, 167, 179, 227, 263, 307, \ldots\}.$$

When x=496, p=373 which does not fit in either A or B, so that there must be other overlapping sequences which form as x increases.

5. Final Comments

The extended Euler-prime-generating equation is shown to be compatible with the composite grid of the modular ring Z_6. x values which yield composite integers belong to sequences within the grid and are associated with (p,t) couples. The primes of the couples are elements of simple sequences. These results illustrate an underlying order of prime formation.

The difference equations in the text can be solved, both analytically and numerically, by a technique based on the work of Asveld [1].

References

1. A.F. Horadam & A.G. Shannon, Asveld's Polynomials. In A.N. Philippou, A.F. Horadam & G.E. Bergum (eds), *Applications of Fibonacci Numbers*. Kluwer: Dordrecht, 1988, pp.163-176.
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AMS Classification Numbers: 11A41, 11A07