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# SOME REPRESENTATIONS CONCERNING THE PRODUCT OF DIVISORS OF n

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Let us denote by $\tau(n)$ the number of all divisors of $n$. It is well-known (see, e.g., [1]) that

$$P_d(n) = \sqrt{n^{\tau(n)}} \qquad (1)$$

and of course, we have

$$p_d(n) = \frac{P_d(n)}{n}. \qquad (2)$$

But (1) is not a good formula for $P_d(n)$, because it depends on function $\tau$ and to express $\tau(n)$ we need the prime number factorization of $n$.

Below, we give other representations of $P_d(n)$ and $p_d(n)$, which do not use the prime number factorization of $n$.

**Proposition 1.** For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \qquad (3)$$

holds.

**Proof.** We have

$$\theta(n,k) \equiv [\frac{n}{k}] - [\frac{n-1}{k}]$$

$$= \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases} \qquad (4)$$

Therefore,

$$\prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} = \prod_{k/n} k \equiv P_d(n)$$

and Proposition 1 is proved.

Here and further the symbols

$$\prod_{k/n} \bullet \text{ and } \sum_{k/n} \bullet$$

mean the product and the sum, respectively, of all divisors of $n$.

The following assertion is obtained as a corollary of (2) and (3).

**Proposition 2.** For $n \geq 1$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{5}$$

holds.

For $n = 1$ we have

$$p_d(1) = 1.$$

**Proposition 3.** For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{6}$$

holds, where here and further we assume that $0! = 1$.

**Proof.** Obviously, we have

$$\frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} = \begin{cases} \frac{n}{k}, & \text{if } k \text{ is a divisor of } n \\ 1, & \text{otherwise} \end{cases}.$$

Hence

$$\prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} = \prod_{k/n} \frac{n}{k} = \prod_{k/n} k \equiv P_d(n),$$

since, if $k$ describes all divisors of $n$, then $\frac{n}{k}$ describes them, too.

Now (2) and (6) yield.

**Proposition 4.** For $n \geq 2$ representation

$$p_d(n) = \prod_{k=2}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{7}$$

holds.

Another type of representation of $p_d(n)$ is the following

**Proposition 5.** For $n \geq 3$ representation

$$p_d(n) = \prod_{k=1}^{n-2} (k!)^{\theta(n,k)-\theta(n,k+1)}, \tag{8}$$

where $\theta(n, k)$ is given by (4).

**Proof.** Let

$$r(n, k) = \theta(n, k) - \theta(n, k + 1).$$

The assertion holds from the fact, that

$$r(n,k) = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \text{ and} \\ & \qquad k+1 \text{ is not a divisor of } n \\ -1, & \text{if } k \text{ is not a divisor of } n \text{ and} \\ & \qquad k+1 \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

We are ready to prove the following interesting

**Theorem.** For $n \geq 2$ the identity

$$\prod_{k=2}^{n} [\frac{n}{k}]! = \prod_{k=1}^{n-1} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{9}$$

holds.

**Proof.** By induction. For $n = 2$ (9) is true. Let us assume, that (9) holds for some $n \geq 2$. Then we must prove that

$$\prod_{k=2}^{n+1} [\frac{n+1}{k}]! = \prod_{k=1}^{n} (k!)^{[\frac{n+1}{k}]-[\frac{n+1}{k+1}]} \tag{10}$$

holds, too.

Dividing (10) by (9) we obtain

$$\prod_{k=2}^{n} \frac{[\frac{n+1}{k}]!}{[\frac{n}{k}]!} = \prod_{k=1}^{n-1} (k!)^{r(n+1,k)}. \tag{11}$$

Sinse, for $k = n+1$

$$[\frac{n+1}{k}]! = 1$$

and for k = n

$$[\frac{n+1}{k}] - [\frac{n+1}{k+1}] = 0.$$

Then (10) is true, if and only if (11) is true. Therefore, we must prove (11) for proving of the Theorem.

From (7), the left hand-side of (11) is equal to $p_d(n+1)$. From (8), the right side of (11) is equal to $p_d(n+1)$, too. Thetrefore, (11) is true.

**Reference**

[1] Nagell T., *Introduction to Number Theory.* John Wiley & Sons, Inc., New York, 1950.