NNTDM 10 (2004), 1, 12-23

THE STRUCTURE OF FIBONACCI NUMBERS IN MODULAR RINGS

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ABSTRACT

An analysis is made of the class and row structures of Fibonacci numbers within the modular ring Z_4. It is found that the class structure repeats the pattern $\bar{0}_4\,\bar{1}_4\,\bar{1}_4\bar{2}_4\bar{3}_4\,\bar{1}_4$. Two thirds of the rows in the ring array are even and all are a sum of Fibonacci numbers. Sums of Fibonacci numbers, covering ten, five and three consecutive numbers or number types, had factors of 11, 11×31, or 101; (these include specific sets). The Fibonacci number primes all belong to the Class $\bar{1}_4$ and therefore equal a sum of squares. There is only one unique set of squares with no common factors. The factors found for the sums have a link with Fermat and Mersenne numbers.

1. INTRODUCTION

The sequence of Fibonacci numbers, $\{F_n\}$, can be defined by the second-order homogeneous linear recurrence relation

$$F_{n+2} = F_{n+1} + F_n,\ n \geq 0, \tag{1.1}$$

with initial terms $F_0 = 0,\ F_1 = 1$. The Binet form of the general terms is given by

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right), \tag{1.2}$$

where α, β are the roots of the associated characteristic equation [1]. We note that

$$\lim_{n\to\infty}\left(\frac{F_n}{F_{n-1}}\right) = \alpha,$$

the golden ratio. Both the Fibonacci numbers and the golden ratio have appeared in the mathematical and scientific literature for hundreds of years [9]. Here we explore the broader characteristics of the Fibonacci numbers in the context of the modular ring Z_4. This is one of a number of such rings which permit new insights into old results as in [6,7].

An integer $n \in Z_4$ can be specified by $(4r_i + i)$ where $\bar{i}$ is the class and r_i the row in a tabular array of the class. Obviously, even integers $\in (\bar{0}_4, \bar{2}_4)$ and odd integers $\in (\bar{1}_4, \bar{3}_4)$ [5,6]. There are no powers in $\bar{2}_4$ and no even powers in $\bar{3}_4$.

2. CLASS AND ROW STRUCTURE OF FIBONACCI NUMBERS IN z_4

Table 1 lists structure for the first 24 Fibonacci numbers. As can be seen, the class structure has the row pattern

$$\bar{0}_4\,\bar{1}_4\,\bar{1}_4\bar{2}_4\bar{3}_4\,\bar{1}_4\,.\;\;\bar{0}_4\,\bar{1}_4\,\bar{1}_4\bar{2}_4\bar{3}_4\,\bar{1}_4\,.\;\ldots$$

which illustrates the dominance of the class $\bar{1}_4$.

n	F_n	**Class**	**Row,** r_n
0	0	$\bar{0}_4$	0
1	1	$\bar{1}_4$	0
2	1	$\bar{1}_4$	0
3	2	$\bar{2}_4$	0
4	3	$\bar{3}_4$	0
5	5	$\bar{1}_4$	1
6	8	$\bar{0}_4$	2
7	13	$\bar{1}_4$	3
8	21	$\bar{1}_4$	5
9	34	$\bar{2}_4$	8
10	55	$\bar{3}_4$	13
11	89	$\bar{1}_4$	22
12	144	$\bar{0}_4$	36
13	233	$\bar{1}_4$	58
14	377	$\bar{1}_4$	94
15	610	$\bar{2}_4$	152
16	987	$\bar{3}_4$	246
17	1597	$\bar{1}_4$	399
18	2584	$\bar{0}_4$	646
19	4181	$\bar{1}_4$	1045
20	6765	$\bar{1}_4$	1691
21	10946	$\bar{2}_4$	2736
22	17711	$\bar{3}_4$	4427
23	28657	$\bar{1}_4$	7164

Table 1: Structure for the first 24 Fibonacci numbers

The row pattern in the final column of Table 1 follows from Equation (1.1) and the structure of the ring; that is, if we define $F_n = 0$ for $n < 0$, then the row r_n is given by

$$r_n = \sum_{j=0} F'_{n-3-6j}, \tag{2.1}$$

in which

$$F'_n = \begin{cases} 0, & n \le 1, \\ F_n, & n > 1. \end{cases}$$

Thus it is only necessary to consider the structure of the first few Fibonacci numbers. That is, with

$$F_n = 4r_i + i,$$

Then starting with n=2, we get

$$F_2 = \quad (4r_0) + (4r_1 + 1)_4. \qquad = 4(r_0 + r_1) + 1 \qquad \in \bar{1}_4 \tag{2.2}$$
$$F_3 = \quad (4r'_1 + 1) + (4r_1 + 1)_4 \qquad = 4(r'_1 + r_1) + 2 \qquad \in \bar{2}_4 \tag{2.3}$$
$$F_4 = \quad (4r''_1 + 1) + (4r_2 + 2)_4 \qquad = 4(r''_1 + r_2) + 3 \qquad \in \bar{3}_4 \tag{2.4}$$
$$F_5 = \quad (4r_3 + 3) + (4r_2 + 2)_4 \qquad = 4(r_3 + r_2 + 1) + 1 \qquad \in \bar{1}_4 \tag{2.5}$$
$$F_6 = \quad (4r'''_1 + 1) + (4r_3 + 3)_4 \qquad = 4(r'''_1 + r_3 + 1) \qquad \in \bar{0}_4 \tag{2.6}$$

and so on. The parity of each row depends on the Class of F_n and the Class of n (Table 2). Note that the $F_n \in \bar{1}_4$ that follows $\bar{3}_4$ always has an odd n.

Class of F_n	**Parity of row**	**Comments**
$\bar{0}_4$	E	always
$\bar{1}_4$	E	$n \in \bar{1}_4$
$\bar{1}_4$	E	$n \in \bar{2}_4$
$\bar{2}_4$	E	always
$\bar{3}_4$	E	$n \in \bar{0}_4$
$\bar{1}_4$	O	$n \in \bar{1}_4$
$\bar{0}_4$	E	always
$\bar{1}_4$	O	$n \in \bar{3}_4$
$\bar{1}_4$	O	$n \in \bar{0}_4$
$\bar{2}_4$	E	always
$\bar{3}_4$	O	$n \in \bar{2}_4$
$\bar{1}_4$	E	$n \in \bar{3}_4$

Table 2: Parities of the rows for the Class of F_n (E: even; O: odd)

3. SUMS OF FIBONACCI NUMBERS

Sets of ten

One of the factors of the sum of ten consecutive Fibonacci numbers is always 11. This follows from mathematical induction and Equation (8) of Horadam [3]:

$$F_{n+i} = F_i F_{n+1} + F_{i-1} F_n,\ i > 1, \tag{3.1}$$

so that

$$\begin{aligned}\sum_{i=0}^{9} F_{n+i} &= F_{n+1}\left(1+\sum_{j=2}^{9} F_j\right) + F_n\left(1+\sum_{k=1}^{8} F_k\right)\\ &= 88F_{n+1} + 55F_n\\ &= 11(8F_{n+1} + 5F_n)\\ &= 11F_{n+6}.\end{aligned} \tag{3.2}$$

We have recently shown [8] that when n is odd {n>0), 11 is always a factor of (10^n+1), while for an even exponent $m(m \in \bar{2}_4)$, the quantity (10^m+1) always has a factor of 101. As shown below, sums of three Fibonacci numbers have a factor of 101. These results show a link among Mersenne, Fermat and Fibonacci numbers.

Sets of five

11 is also a factor of the sum of five consecutive odd-subscripted Fibonacci numbers; that is, with n odd:

$$\sum_{i=0}^{4} F_{n+2i} = 11F_{n+4}. \tag{3.3}$$

This follows from the lacunary recurrence relations in [10]:

$$\begin{aligned}\sum_{i=0}^{4} F_{n+2i} &= F_n + F_{n+2} + F_{n+4} + F_{n+6} + F_{n+8}\\ &= 22F_n + 33F_{n+1},\end{aligned} \tag{3.4}$$

so that

$$S = \sum_{i=0}^{4} F_{n+2i} = 11(2F_n + 3F_{n+1}) = 11F_{n+4}. \tag{3.5}$$

In the modular ring Z_4, a set of five consecutive odd-sub-scripted Fibonacci numbers must have one of the following class structures:

I $\bar{1}_4\ \bar{2}_4\ \bar{1}_4\ \bar{1}_4\ \bar{2}_4$

II $\bar{2}_4\ \bar{1}_4\ \bar{1}_4\ \bar{2}_4\ \bar{1}_4$

$$\text{III} \quad \bar{1}_4 \bar{1}_4 \bar{2}_4 \bar{1}_4 \bar{1}_4$$

This is because both the Classes $\bar{3}_4$ and $\bar{0}_4$ always have even *n*. If we take the Set I above, with the rows represented by r_i, then

$$S = 4\left(r_1 + r_2 + r_1^{/} + r_1^{//} + r_2^{/} + 1\right) + 3 \tag{3.6}$$

so that $S \in \bar{3}_4$. This will be the case for both Sets I and II. The prime $11 \in \bar{3}_4$ so that the other component must be in $\bar{1}_4$ since $\bar{3}_4 \times \bar{1}_4 \rightarrow \bar{3}_4$ whereas $\bar{3}_4 \times \bar{3}_4 \rightarrow \bar{1}_4$. Set III will have $S \in \bar{2}_4$ so that the other component will be in $\bar{2}_4$ since $\bar{3}_4 \times \bar{2}_4 \rightarrow \bar{2}_4$.

As an example, consider the case when *n*=7 with $F_7 = 13 \in \bar{1}_4$. The associated sum will have the class structure of Set I. (The interested reader might like to try Class II or Class III.)

Using Equations (2.1) and (3.6), we obtain

$$\begin{aligned} \sum r_i &= F_4 + F_6 + F_8 + F_2 + F_{10} + F_4 + F_{12} + F_6 \\ &= 3 + 8 + 21 + 1 + 55 + 3 + 144 + 8 \\ &= 243 \end{aligned} \tag{3.7}$$

so that

$$4\left(\sum r_i + 1\right) + 3 = 979 = 11 \times 89. \tag{3.8}$$

Of course, sums of five consecutive even-subscripted Fibonacci numbers must also have 11 as a factor. This follows from the fact that ten consecutive Fibonacci numbers have 11 as a factor; as do multiples of ten consecutive numbers in general.

Class specific sets

Class $\bar{2}_4$ sum

When five consecutive $\bar{2}_4$ Fibonacci numbers are summed the result has a factor of 11×31 or 341; that is,

$$\sum_{t=0}^{4} F_{n+6t} = \left(11 \times 31\right) F_{n+12}. \tag{3.9}$$

The coefficient 6 occurs because *n*=3+6*q*, *q*=0,1,2,3,…, which yields the *n* values of $F_n \in \bar{2}_4$.

Since each

$$F_{ni} = 4r_{2i} + 2,$$

then

$$\sum_{t=0}^{4} F_{n+6t} = 4\left(\sum_{i=1}^{5}(r_{2i}) + 2\right) + 2, \tag{3.10}$$

so that the sum must also be in Class $\bar{2}_4$. Since $11 \in \bar{3}_4$, $31 \in \bar{3}_4$ and $F_{n+12} \in \bar{2}_4$, the products $\bar{3}_4 \times \bar{3}_4 = \bar{1}_4$ and $\bar{1}_4 \times \bar{2}_4 = \bar{2}_4$. For example,

$$\begin{aligned} \sum_{t=0}^{4} F_{3+6t} &= F_3 + F_9 + F_{15} + F_{21} + F_{27} \\ &= 2 + 34 + 610 + 10946 + 196418 \\ &= (11 \times 31) \times 610 \\ &= (11 \times 31)\, F_{15}. \end{aligned} \tag{3.11}$$

<u>*Class $\bar{0}_4$ sum*</u>.

When five consecutive Class $\bar{0}_4$ Fibonacci numbers are added, the sum has a factor (11×31) as in Class $\bar{2}_4$. Here n=6w, w=1,2,3,… . Hence,

$$\sum_{t=0}^{4} F_{n+6t} = (11 \times 31) F_{n+12} \in \bar{0}_4 \tag{3.12}$$

because $F_{n+12} \in \bar{0}_4$ and $\bar{3}_4 \times \bar{3}_4 \times \bar{0}_4 \to \bar{0}_4$. For instance,

$$\begin{aligned} \sum_{t=0}^{4} F_{6+6t} &= F_6 + F_{12} + F_{18} + F_{24} + F_{30} \\ &= 8 + 144 + 2584 + 46368 + 832040 \\ &= (11 \times 31) \times 2584 \\ &= (11 \times 31) F_{18}. \end{aligned} \tag{3.13}$$

<u>*Class $\bar{3}_4$ sum*</u>

The n values for $F_n \in \bar{3}_4$ are generated by n=4+6v, v=0,1,2,3,… . As for $\bar{2}_4$ and $\bar{0}_4$ sums of five consecutive Fibonacci numbers in $\bar{3}_4$ follow the pattern of Equation (3.9), but the sum must fall in $\bar{3}_4$ because $F_{n+12} \in \bar{3}_4$ and $(11 \times 31) \in \bar{1}_4$. For example,

$$\begin{aligned} \sum_{t=0}^{4} F_{4+6t} &= F_4 + F_{10} + F_{16} + F_{22} + F_{28} \\ &= 3 + 55 + 987 + 17711 + 317811 \\ &= (11 \times 31) \times 987 \\ &= (11 \times 31) F_{16}. \end{aligned} \tag{3.14}$$

Class $\bar{1}_4$ sum

The Fibonacci numbers in this Class are of three types with $=a+6q, a \in \{1,2,5\}$. When five consecutive $F_n \in \bar{1}_4$ are summed, we get (just as for the other three Classes):

$$\sum_{t=0}^{4} F_{n+6t} = (11\times 31)F_{n+12} \in \bar{1}_4$$

because $F_{n+12} \in \bar{1}_4$ and $(11\times 31) \in \bar{1}_4$. As an illustration, consider

$$\begin{aligned}\sum_{t=0}^{4} F_{1+6t} &= F_1 + F_7 + F_{13} + F_{19} + F_{25}\\ &= 1+13+233+4181+75025\\ &= (11\times 31)\times 233\\ &= (11\times 31)F_{13}.\end{aligned} \tag{3.15}$$

The same applies to the other $\bar{1}_4$ types. From Equation (3.1) we have that the sum, *S*, is, after repeated use of Equation (3.1), given by

$$S = F_n + F_{n+6} + F_{n+12} + F_{n+18} + F_{n+24} \tag{3.16}$$

$$\begin{aligned}&= F_n + (F_6F_{n+1} + F_5F_n) + (F_{12}F_{n+1} + F_{11}F_n) +\\ &\qquad (F_{18}F_{n+1} + F_{17}F_n) + (F_{24}F_{n+1} + F_{24}F_{23}F_n)\\ &= 49104F_{n+1} + 30349F_n\\ &= (11\times 31\times 144)F_{n+1} + (11\times 31\times 89)F_n\\ &= (11\times 31)(F_{12}F_{n+1} + F_{11}F_n)\\ &= (11\times 31)F_{n+12}.\end{aligned} \tag{3.17}$$

When the sum of fifteen evenly spaced Fibonacci numbers, all in $\bar{1}_4$, is taken we find that

$$S = (11\times 31)\left(F_{n_1+12} + F_{n_2+12} + F_{n_5+12}\right) \in \bar{3}_4 \tag{3.18}$$

where $n_1 = 1+6q,\ n_2 = 2+6q,\ n_5 = 5+6q,\ q$ constant for the set. For example, with $q=2, n_1 = 13, n_2 = 14, n_5 = 17,$ we have

$$\begin{aligned}S &= (11\times 31)(F_{25} + F_{26} + F_{29})\\ &= (11\times 31)(75025 + 121393 + 514229)\\ &= (11\times 31)(7\times 7\times 14503) \in \bar{3}_4.\end{aligned}$$

Sums with a factor 101

A sum of three Fibonacci numbers yields a factor of 101, but the pattern is more complex than previously (Table 4).

Sum	Class structure of F_n
$F_{11}+F_{13}+F_{17}=101\times19$	$\bar{1}_4+\bar{1}_4+\bar{1}_4$
$F_{11}+F_{13}+F_{33}=101\times2^2\times5^2\times349$	$\bar{1}_4+\bar{1}_4+\bar{2}_4$
$F_{13}+F_{17}+F_{39}=101\times2^3\times78277$	$\bar{1}_4+\bar{1}_4+\bar{2}_4$
$F_{13}+F_{33}+F_{39}=101\times661097$	$\bar{1}_4+\bar{2}_4+\bar{2}_4$

Table 4: Sums of three Fibonacci numbers

4. FIBONACCI PRIMES

It is well known that if F_p is a prime then *p* is prime, but if *p* is prime, then F_p may be composite [9] (Table 5).

n	F_n	**Factors**	**Class of**		**(*d*,*e*)**	**Rows**		**Class**	
			n	F_n		n	F_n	F_{n-1}	F_{n+1}
5	5	*p*	$\bar{1}_4$	$\bar{1}_4$	(2,1)	1	1	$\bar{3}_4$	$\bar{0}_4$
7	13	*p*	$\bar{3}_4$	$\bar{1}_4$	(3,2)	1	3	$\bar{0}_4$	$\bar{1}_4$
11	89	*p*	$\bar{3}_4$	$\bar{1}_4$	(8,5)	2	22	$\bar{3}_4$	$\bar{0}_4$
13	233	*p*	$\bar{1}_4$	$\bar{1}_4$	(13,8)	3	58	$\bar{0}_4$	$\bar{1}_4$
17	1597	*p*	$\bar{1}_4$	$\bar{1}_4$	(21,34)	4	399	$\bar{3}_4$	$\bar{0}_4$
19	4181	37,113	$\bar{3}_4$	$\bar{1}_4$	(55,34) (41,50)	4	1045	$\bar{0}_4$	$\bar{1}_4$
23	28657	p	$\bar{3}_4$	$\bar{1}_4$	(144),(89)	5	7164	$\bar{3}_4$	$\bar{0}_4$
29	514229	p	$\bar{1}_4$	$\bar{1}_4$	(610),(377)	7	128557	$\bar{3}_4$	$\bar{0}_4$
31	1346269	557,2417	$\bar{3}_4$	$\bar{1}_4$	(987,610) (875,762)	7	336567	$\bar{0}_4$	$\bar{1}_4$
37	24157817	73,149,2221	$\bar{1}_4$	$\bar{1}_4$	(4181,2584) (4909,244) (3859,3044)	9	6039454	$\bar{0}_4$	$\bar{1}_4$

Table 5: Fibonacci primes and pseudo-primes [9]

An important feature of Table 5 is that all $F_n \in \bar{1}_4$. Integers in this Class equal a sum of squares $\left(d^2+e^2\right)$, but primes only have a unique pair of (*d*,*e*) with no common factors [7]. Composites have the same number of (*d*,*e*) pairs as they have factors. Thus, all Fibonacci numbers with a prime subscript can be sieved out and then checked for (*d*,*e*) pairs. The *d* and *e* for primes are Fibonacci numbers and are simply obtained [7] from

$$d = F_{\frac{1}{2}(n+1)},\ e = F_{\frac{1}{2}(n-1)}.$$

Row and Class patterns for F_n and n could also be explored. In Table 5, the composites occur for repeat rows, for example. It is worth investigating why some sets of five Fibonacci numbers with prime subscripts have a factor of 13 and others do not; for instance,

$$13 | (F_5 + F_7 + F_{11} + F_{13} + F_{17})$$

and

$$13 | (F_7 + F_{11} + F_{13} + F_{17} + F_{23}),$$

but

$$13 \nmid (F_{11} + F_{13} + F_{17} + F_{23} + F_{29}).$$

Furthermore, one could look at the patterns of row nesting; that is, the rows of the rows of the rows… or Meta-Fibonacci sequences [12]).

5. SUMS OF PRODUCTS OF TWO CONSECUTIVE FIBONACCI NUMBERS

It is well known that [9]

$$\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2. \tag{5.1}$$

Equation (5.1) can be generalized as follows:

$$\sum_{k=1}^{j-1} F_{i+k} F_{i+k+1} = F_{j+1}^2 - F_{i+1}^2 \tag{5.2}$$

with i and j having the same parity. It then follows from Equation (5.2) that

$$F_j^2 - F_i^2 = F_{j-i} F_{j+i}, \tag{5.3}$$

which is a generalization of the known Identities $(I_{25}),(I_{26})$ of [2], namely,

$$F_{n+p}^2 - F_{n-p}^2 = F_{2n} F_{2p},$$
$$F_{n+1}^2 - F_{n-2}^2 = 4F_n F_{n-1}.$$

The question arises whether the difference of squares in Equation (5.2) could equal a square or some other power? Some examples of Equations (5.2) and (5.3) are shown in Table 6.

i	***j***	**Sum**	***j-i***	$F_{j+2}^2 - F_{i+2}^2$	$F_{j-i} F_{j+i+4}$
0	6	440	6	$21^2 - 1^2 = 440$	$8 \times 55 = 440$
1	17	17480757	16	17480761-4=17480757	$987 \times 17711 = 17480757$
2	12	142120	10	$377^2 - 3^2 = 142120$	$55 \times 2584 = 142120$

5	19	119814747	14	$10946^2 - 13^2 = 119814747$	$377 \times 317811 = 119814747$

Table 6: Examples of Equations (5.2) and (5.3)

For an application of the sequence $\{F_n F_{n+1}\}$ in Equation (5.1) see [1]. Little is recorded in the literature about this sequence. If we let

$$U_n = F_n F_{n+1},$$

then it can be established that $\{U_n\}$ satisfies the second-order inhomogeneous recurrence relation

$$U_n = U_{n-1} + U_{n-2} + 2F_{n-1}^2 - (-1)^n,\ n > 2, \tag{5.4}$$

with initial conditions $U_1 = 1, U_2 = 2.$

Proof:

$$\begin{aligned} U_n &= (F_{n-1} + F_{n-2})(F_n + F_{n-1}) \\ &= U_{n-1} + U_{n-2} + F_{n-1}^2 + F_n F_{n-2} \end{aligned}$$

from which the result follows from the use of Simson's identity:

$$F_n F_{n-2} - F_{n-1}^2 = (-1)^n. \qquad \blacksquare$$

6. FINAL COMMENTS

Row zero in Z_4 has integers with right end digits (REDs) (indicated by an asterisk) of 0,1,2,3. This is repeated in the fifth row. The first row has REDs of 4,5,6,7, as does the sixth row. This regular pattern for the REDs gives rise to periodicity in the REDs of Fibonacci numbers. For example, numbers that fall in Class $\bar{1}_4$ with $F_n^* = 1,$ always fall in a row with a RED of 0 or 5. Some examples of this are set out in Table 7.

F_n	**Rows**	**REDs**
F_1	0	0
F_2	0	0
F_8	5	5
F_{19}	1045	5
F_{41}	41395035	5
F_{59}	239180506510	0
F_{61}	626182695490	0
F_{62}	1013184884470	0
F_{68}	18180865062035	5

F_{79}	3618083506169055	5

Table 7: Right End Digit Patterns

A periodicity of 60 has been noted for $F_n^* = 1$; when n=2 and 62, for instance [9]. However, it is more instructive to look at periodicity in terms of the integer structure. When $F_n^* = 1,\ F_n \in \bar{3}_4$, the rows r_3 have REDs of 2 or 7. For example, the REDs of the rows for $F_{22}, F_{28}, F_{82}, F_{88}$ are 7,2,7,2, respectively, which shows the interval of 5 as expected. Note that the periodicity of 60 corresponds to the same RED for the rows. In general, any Fibonacci number will always have rows such that the intervals for the REDs are 0 or 5; the Fibonacci numbers must be in the same Class (Table 8).

F_n^*	r_1^*	r_3^*	F_n^*	r_0^*	r_2^*
1	0,5	2,7	0	0,5	2,7
3	3,8	0,5	2	3,8	0,5
5	1,6	3,8	4	1,6	3,8
7	4,9	1,6	6	4,9	1,6
9	2,7	4,9	8	2,7	4,9

Table 8: $F_n = 4r_i + i,\ i = 0,1,2,3$

The sequences for the last 2 digits (or 3 digits or n digits) may be analysed similarly. For instance, using $r_i^* = 01,02,03,04,05,06,07,08,09,00,\ i \in \{1,3\}$, we can deduce that $F_n \in \bar{1}_4$ when the last two digits are 01,05,09,13,17,21,25,29,33,37, whereas $F_n \in \bar{3}_4$ if the last two digits are 03,07,11,15,19,23,27,31,35 or 39.

Other interesting Fibonacci number results could be analysed with modular rings, such as Horadam's Pythagorean number triples [4].

REFERENCES

1. Brousseau, Brother Alfred. 1968. A Sequence of Power Formulas. *The Fibonacci Quarterly*. **6.1**:81-83.
2. Hoggatt, Verner E. Jr. 1969 *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin.
3. Horadam, AF. 1961. A Generalized Fibonacci Sequence. *American Mathematical Monthly*. **68.5**: 455-459.
4. Horadam, A.F. 1961. Fibonacci Number Triples. *American Mathematical Monthly*. **68.8**: 751-753.
5. Leyendekkers, J.V., Rybak, J.M. & Shannon, A.G. 1995. Integer Class Properties Associated with an Integer Matrix. *Notes on Number Theory & Discrete Mathematics*. **1.2**: 53-59.
6. Leyendekkers, J.V., Rybak, J.M. & Shannon, A.G. 1997. Analysis of Diophantine Properties Using Modular Rings with Four and Six Classes. *Notes on Number Theory & Discrete Mathematics*. **3.2**: 61-74.

7. Leyendekkers, J.V., & Shannon, A.G. 1998. Fibonacci Numbers within Modular Rings. *Notes on Number Theory & Discrete Mathematics.* **4.4**: 165-174.

8. Leyendekkers, J.V., & Shannon, A.G. 1999. Fermat and Mersenne Numbers and Some Related Factors. *International Journal of Mathematical Education in Science & Technology*. **30.4**: 627-629.

9. Livio, Mario. 2002. *The Golden Ratio.* New York: Broadway Books.

10. Rotkiewicz, Andrzej. 1999. Some Unsolved Problems on Pseudoprime Numbers and Their Generalizations. In Fredric T Howard (ed.), *Applications of Fibonacci Numbers, Volume 8*. Dordrecht: Kluwer, pp.293-306.

11. Shannon, A.G., & Horadam, A.F. 1979. Special Recurrence Relations Associated with the Sequence $\{w_n(a,b;p,q)\}$. *The Fibonacci Quarterly*. **17.4**: 294-299.

12. Vajda, S. 1989. *Fibonacci and Lucas Numbers, and the Golden Section*. Chichester: Ellis Horwood, Ch.XII.

AMS Classification Numbers: 06F25, 11B39