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THE ROW STRUCTURE OF SQUARES IN MODULAR RINGS

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Abstract

Modular-ring row structures are developed for squares. In particular, the row structures of even squares within the modular ring Z_6 are analysed. This structure is shown to be linked via generalized pentagonal numbers to that of the odd squares. When $3 \mid N$, the link is via the triangular numbers. Equations could thus be developed for the rows of those primes that equal a sum of squares. Since the results are general they can be used to study in some depth those systems that have squares as a dominant feature.

1. Introduction

The study of squares has been going on since antiquity. All powers equal a difference of squares [1], and the sum of the squares has been intensively studied. For example, Pythagoras' Theorem, and Fermat's observation that primes in Class $\bar{1}_4$ of the Modular Ring Z_4 equal a sum of squares [2].

The squares of odd integers are well characterised within this ring, in which the integers can be represented by $(4r_i + i)$, where $\bar{i}$ is the class and *r* the row in a modular array. In Z_4, the square of an odd integer, *N*, $(3 \nmid N)$, is given by [1]

$$N^2 = 4(6K) + 1 \tag{1.1}$$

where

$$K_i = \begin{cases} \tfrac{1}{2} n(3n-1), & even\, i, (n = n_1), \\ \tfrac{1}{2} n(3n+1), & odd\, i, \, (n = n_2). \end{cases} \tag{1.2}$$

Here, K_i is a generalized pentagonal number.
If $3|N$, then the row of the square, R_1, is given by

$$R_1 = 2 + 18\sum_{i=1}^{j} i = 2 + 18\left(\tfrac{1}{2} j(j+1)\right) \tag{1.3}$$

with

$$j = \tfrac{1}{6}(N-3),$$

and $\tfrac{1}{2} j(j+1)$ are the triangular numbers.
For the modular ring Z_6 (where $N = 6r_i + (i-3)$),

$$N^2 = 6(4K) + 1, \tag{1.4}$$

except when 3|*N*; for this case, see Section 2, Class $\bar{3}_6$.
We have recently shown [3] that when $n = n_1$, then $N \in \bar{2}_6$, $(N = 6r_2 - 1)$, but if $n = n_2$, then $N \in \bar{4}_6$, $(N = 6r_4 + 1)$. Of course, just as for Z_4, so too in Z_6 one of the classes of odd integers does not contain any even exponents, that is Class $\bar{3}_4 \in Z_4$ and $\bar{2}_6 \in Z_6$.
Thus while the rows of odd squares are well characterised, this is not the case for the even integers. Our first task here, then, is to analyse the rows of even integers. We shall confine our study to Z_6 since the results can be readily applied to other modular rings.

2. Squares of Even Integers in Z_6

2.1 Class $\bar{1}_6$

In this Class,

$$N = (6r_1 - 2) \in \bar{1}_6.$$

r_1 ***even***.
When the row is even, ½*N* is odd, so that *N*=2*Q* where *Q* is odd and falls in $\bar{2}_6$. Hence, as odd squares only fall in $\bar{4}_6$.

$$\tfrac{1}{4}N^2 = 6(4K) + 1, \tag{2.1}$$

and

$$N^2 = 6(16K + 1) - 2, \tag{2.2}$$

so that the row of the Square, R_1, equals (16*K*+1), with

$$K = \tfrac{1}{2}n_1(3n_1 - 1),$$

as $Q \in \bar{2}_6$.

r_1 ***odd***.
In this case

$$N = 2^m Q,\ m = 2,3,4,5,.... \tag{2.3}$$

The row functions depend on the parity of *m* but, in general,

$$r_1 = A + 2^m t,\ t = 0,1,2,3,..., \tag{2.4}$$

while the functions for *A* are listed in Table 1.

Class	*m*	*A*	Class of *Q*
$\bar{1}_6$	even odd	$\tfrac{1}{6}(2^m + 2)$ $\tfrac{1}{6}(5 \times 2^m + 2)$	$\bar{4}_6$ $\bar{2}_6$
$\bar{5}_6$	even odd	$\tfrac{1}{6}(5 \times 2^m - 2)$ $\tfrac{1}{6}(2^m - 2)$	$\bar{2}_6$ $\bar{4}_6$

Table 1: Functions for rows of integers in $\bar{1}_6$ and $\bar{5}_6$ (Equation (2.4))

From Equation (2.3)

$$N^2 = 2^{2m} Q^2 \tag{2.5}$$

and since the squares all fall in Class $\bar{1}_6$

$$N^2 = 6R_1 - 2 \tag{2.6}$$

with R_1 the row of the square. Hence,

$$R_1 = \tfrac{1}{6}\left(2^{2m} Q^2 + 2\right). \tag{2.7}$$

Furthermore, since $N = 6r_1 - 2$, then

$$N^2 = 36r_1^2 - 24r_1 + 4, \tag{2.8}$$

so that

$$R_1 = 6r_1^2 - 4r_1 + 1 \tag{2.9}$$

with r_1 given by Equation (2.4) and Table 1.

m even

For this case, with $Q \in \bar{4}_6$ (Table 1),

$$N^2 = 2^{2m}\left(6r_4 + 1\right)^2 \tag{2.10}$$

with r_4 the row of $Q \in \bar{4}_6$ (Table 1). Then, since the row of this odd integer squared is 4*K*, (Equation (1.4)),

$$N^2 = 2^{2m}\left(6(4K) + 1\right) \tag{2.11}$$

with

$$K = \tfrac{1}{2} n_2 (3n_2 + 1),$$

so that

$$R_1 = 2^{2m}\left(4K + (1/6)\right) + (1/3) \tag{2.12}$$

and

$$R_1 = 2^{2m}\left(6n_2^2 + 2n_2 + (1/6)\right) + (1/3) \tag{2.13}$$

with n_2 being the row of $Q \in \bar{4}_6$.

Alternatively, using Equation (2.9) and Table 1, we get

$$R_1 = 2^{2m}\left(6t^2 + 2t + (1/6)\right) + (1/3). \tag{2.14}$$

Hence,

$$n_2 = t.$$

m odd

Similarly,

$$N^2 = 2^{2m}\left(6r_2 - 1\right)^2,$$

and

$$N^2 = 6R_1 - 2 \tag{2.15}$$

where r_2 is the row of $Q \in \bar{2}_6$ and R_1 the row of $N^2 \in \bar{1}_6$. Since $\bar{2}_6$ has no squares and

$$\left(6r_2 - 1\right)^2 = 6(4K') + 1,$$

then

$$R_1 = 2^{2m}\left(4K' + (1/6)\right) + (1/3) \tag{2.16}$$

with

$$K' = \tfrac{1}{2} n_1 (3n_1 - 1),$$

as $Q \in \bar{2}_6$. Thus, with n_1 the row of $Q \in \bar{2}_6$,

$$R_1 = 2^{2m}\left(6n_1^2 - 2n_1 + (1/6)\right) + (1/3). \tag{2.17}$$

Using Equation (2.9) and Table 1, and comparing the R_1 function with that obtained in Equation (2.17), we obtain

$$n_1 = t + 1.$$

Some examples are given in Table 2.

N	r_1	Q	Class of Q	m	A	t	n	R_1
28	5	7	$\bar{4}_6$	2	1	1	1	131
88	15	11	$\bar{2}_6$	3	7	1	2	1291
172	29	43	$\bar{4}_6$	2	1	7	7	4931
544	91	17	$\bar{2}_6$	5	27	2	3	49323
6400	1067	25	$\bar{4}_6$	8	43	4	4	6826667

Table 2: Examples of Equations (2.13) and (2.17)

2.2 Class $\bar{5}_6$

In this Class,

$$N = 6r_5 + 2.$$

When integers from this Class are squared, the resultant lies in Class $\bar{1}_6$, since $\bar{5}_6$ contains no even powers.

r_5 ***even***

As for $\bar{1}_6$, when r_5 is even,

$$N=2Q,$$

where Q is odd but falls in $\bar{4}_6$, in this case, with a row equal to $4K$. Hence,

$$N^2 = 6(16K + 1) - 2 \tag{2.18}$$

and

$$R_1 = 16K + 1,$$

with

$$K = \tfrac{1}{2} n_2 (3n_2 + 1),$$

since $Q \in \bar{4}_6$.

r_5 ***odd***

As for $\bar{1}_6$,

$$N = 2^m Q, \tag{2.19}$$

with m=2,3,4,5,…, and

$$N = 6r_5 + 2, \tag{2.20}$$

with

$$r_5 = A + 2^m t. \tag{2.21}$$

The functions for A are listed in Table 1.

$$N^2 = 2^{2m} Q^2 \tag{2.22}$$

and since integers in $\bar{5}_6$, when squared, fall in $\bar{1}_6$,

$$N^2 = 6R_1 - 2 \tag{2.23}$$

so that

$$R_1 = (1/6)(2^{2m} Q^2 + 2). \tag{2.24}$$

Moreover, from Equation (2.20),

$$N^2 = 36r_5^2 + 4r_5 + 1 \tag{2.25}$$

with r_5 from Table 1.

m even

$Q \in \bar{2}_6$ (Table 1) so that

$$N^2 = 2^{2m}(6r_2 - 1)^2 \tag{2.26}$$

with r_2 the row of Q. But

$$(6r_2 - 1)^2 = 6R_4 + 1$$

with

$$R_4 = 4K,$$

so that

$$N^2 = 2^{2m}(6(4K) + 1) \tag{2.27}$$

with

$$K = \tfrac{1}{2} n_1 (2n_1 - 1),$$

so that

$$\begin{aligned} R_1 &= 2^{2m}(4K + (1/6)) + (1/3) \\ &= 2^{2m}\left(6n_1^2 - 2n_1 + (1/6)\right) + (1/3) \end{aligned} \tag{2.28}$$

with n_1 being the row of $Q \in \bar{2}_6$.
Alternatively, using Equation (2.25) and Table 1, we get

$$R_1 = 2^{2m}((6t + 4)(t + 1) + (1/6)) + (1/3). \tag{2.29}$$

When $n_1 = (1 + t)$ is substituted into Equation (2.28), one gets Equation (2.29), which shows that $t = n_1 - 1$.

m odd

Here $Q \in \bar{4}_6$ (Table 1), in row r_4, so that

$$N^2 = 2^{2m}(6r_4+1)^2$$
$$= 6R_1 - 2, \tag{2.30}$$

with

$$(6r_4+1)^2 = 6(4K^{/})+1, \tag{2.31}$$

so that

$$R_1 = 2^{2m}\left(4K^{/} + (1/6)\right) + (1/3) \tag{2.32}$$

with

$$K^{/} = \tfrac{1}{2} n_2 (3n_2+1),$$

as $Q \in \overline{4}_6$. Thus

$$R_1 = 2^{2m}\left(6n_2^2 + 2n_2 + (1/6)\right) + (1/3) \tag{2.33}$$

with n_2 being the row of $Q \in \overline{4}_6$, that is, r_4. Using Equation (2.25) and Table 1, we get

$$R_1 = 2^{2m}\left(6t^2 + 2t + (1/6)\right) + (1/3) \tag{2.34}$$

which shows that $t = n_2$ in this case.

Class $\overline{3}_6$

In this Class,

$$N = 6r_3$$

and obviously $N^2 \in \overline{3}_6$, that is,

$$N^2 = 6R_3.$$

r_3 ***odd***

Unlike Classes $\overline{1}_6, \overline{5}_6$, ½*N* will be odd when *N* is in an odd numbered row, and

$$N=2Q \tag{2.35}$$

where 3|*Q*, so that *Q* always falls in Class $\overline{6}_6$, that is

$$Q = 6r_6 + 3.$$

The row of a square in $\overline{6}_6, R_6$, is given by

$$R_6 = 1 + 12\sum_{i=1}^{j} i \tag{2.36}$$

from the Z_4 row (Equation (1.3)). Thus, with

$$N^2 = 4Q^2 = 24R_6 + 12 = 6R_3, \tag{2.37}$$

so that

$$R_3 = 6\left(1 + 8\sum_{i=1}^{j} i\right) \tag{2.38}$$

with

$$\sum_{i=1}^{j} i = \tfrac{1}{2} j(j+1)$$

where i refers to Q_i (for example, the (N,i) sequence for $\overline{6}_6$ is {(3,0), (9,1), (15,2), (21,3), …). In effect, $j \equiv r_6$. Note that this is in contrast to Z_4 where integers 3|N have $j = \frac{1}{6}(N-3)$, because $N_i - N_{i-1} = 6$.

r_3 ***even***

In this case,

$$N = 2^m Q, \tag{2.39}$$

and

$$N^2 = 2^{2m} Q^2 = 2^{2m}(6R_6 + 3) = 6R_3. \tag{2.40}$$

Thus

$$\begin{aligned} R_3 &= 2^{2m-1} \times 3\left(1 + 8\sum_{i=1}^{j} i\right) \\ &= 2^{2m-1} \times 3(1 + 4j(j+1)). \end{aligned} \tag{2.41}$$

When m=1, Equation (2.41) becomes Equation (2.38). The results for this Class are much simpler than those for $\overline{1}_6$ and $\overline{5}_6$. This is because

- Q falls in only one class,
- 3 is a common factor, and
- the term $(6r_i + (i-3))$ has i=3 so that there is no constant.

3. Primes as a Sum of Squares

Primes, p, which fall in $\overline{1}_4 \in Z_4$ (that is, $p = 4r_1 + 1$) equal a unique sum of squares, $(x^2 + y^2)$, where (x,y)=1 [2]. Fermat appears to have been the first to have established this. In Z_6, these primes fall in $\overline{2}_6$ in odd rows or in $\overline{4}_6$ in even rows [3] (Table 3).

p	Class in Z_4	r_4	Class in Z_6	r_i	i
5	$\overline{1}_4$	1	$\overline{2}_6$	1	2
13	$\overline{1}_4$	3	$\overline{4}_6$	2	4
17	$\overline{1}_4$	4	$\overline{2}_6$	3	2
29	$\overline{1}_4$	7	$\overline{2}_6$	5	2
37	$\overline{1}_4$	9	$\overline{4}_6$	6	4

Table 3: Examples of $p = x^2 + y^2$

Consider the primes in Class $\overline{2}_6$ in odd rows. There will be certain constraints on the Classes of x and y (Table 4).

p	x	y	x^2	y^2
$\bar{2}_6$	$\bar{2}_6$	$\bar{1}_6$	$\bar{4}_6$	$\bar{1}_6$
	$\bar{2}_6$	$\bar{5}_6$	$\bar{4}_6$	$\bar{1}_6$
	$\bar{4}_6$	$\bar{1}_6$	$\bar{4}_6$	$\bar{1}_6$
	$\bar{4}_6$	$\bar{5}_6$	$\bar{4}_6$	$\bar{1}_6$

Table 4: Class structures

However, all the odd squares will fall in $\bar{4}_6$ and the even squares in $\bar{1}_6$. Thus $(x^2 + y^2)$ or p, will fall in the row $(R_1 + R_4) \in \bar{2}_6$, where R_1 is the row of y^2 and R_4 is the row of x^2. On the other hand, primes in $\bar{4}_6$ in even rows will have (x^2, y^2) pairs in Classes $(\bar{4}_6, \bar{3}_6)$ and $(\bar{6}_6, \bar{1}_6)$ with the corresponding rows for p of $(R_4 + R_3)$ and $(R_6 + R_1)$.

For primes in $\bar{2}_6$ in odd rows, and with $x \in \bar{2}_6$ in row n_1,

$$R_4 = 4K = 2n_1(3n_1 - 1).$$

With the row of y even, $R_1 = (16K + 1)$ where $K = f(n_1')$ when $y \in \bar{1}_6$ or $K = f(n_2')$ when $y \in \bar{5}_6$ (Section 2). Thus, in the simplest case, when y is in an even row, r_p, the row of the prime in $\bar{2}_6$, is given by

$$r_p = \begin{cases} 2n_1(3n_1 - 1) + 8n_1'(3n_1' - 1) + 1, & y \in \bar{1}_6, \\ 2n_1(3n_1 - 1) + 8n_2'(3n_2' + 1) + 1, & y \in \bar{5}_6. \end{cases} \tag{3.1}$$

These rows are obviously odd.

When $x \in \bar{4}_6, K = \frac{1}{2} n_2(3n_2 + 1)$ and the first term on the right hand side of Equation (3.1) becomes $2n_2(3n_2 + 1)$. When the row of y is odd, Equations (2.13) and (2.17) give R_1 for $y \in \bar{1}_6$, while Equations (2.28) and (2.33) give R_1 for $y \in \bar{5}_6$.

Examples are given in Table 5, with x=5, $\bar{2}_6, n_1 = 1$.

Q=½y	Class of ½y	$n_1^/$	r_p	Prime ($6r_p - 1$)
11	$\bar{2}_6$	2	85	509
17		3	197	1181
23		4	357	2141
29		5	565	3389
47		8	1477	8861
53		9	1877	11261
77		13	3957	23741
83		14	4597	27581
101		17	6805	40829
107		18	7637	45821
131		22	11445	68669

Table 5(a): Examples of Equation (3.1) [y in even row in $\bar{1}_6$]

Q=½y	Class of ½y	$n_2^/$	r_p	Prime ($6r_p - 1$)
1	$\bar{4}_6$	0	5	29
13		2	117	701
37		6	917	5501
49		8	1605	9629
67		11	2997	17981
73		12	3557	21341
79		13	4165	24989
91		15	5525	33149
103		17	7077	42461
151		25	15205	91229
157		26	16437	98621

Table 5(b): Examples of Equation (3.1) [y in even row in $\bar{5}_6$]

It is of interest that, within the range of Table 5, ½y is always a prime or 7|½y, while the right end digits (REDs) are very restricted, that is $\left(r_p^*, p^*\right) \in \{(5,9),(7,1)\}$.

The $n_1^/, n_2^/$ values in Table 5 are sequential, but the rules of formation are not yet known. Table 6 gives examples of row sequences for various values of $x \in \bar{2}_6$. There are generally more members for $n_2^/$ than for $n_1^/$. Obviously, if the recurrence relations for these sequences could be defined, then the associated primes could be directly predicted from $\left(6r_p - 1\right)$. The same applies for Class $\bar{4}_6$. For example, when x=5, $n_1^/$ as a j sequence follows

$$\alpha_i + \beta_{i+1} = \alpha_{i+1} + \beta_i + \omega_k$$

with

$$\omega_k = 0,0,0,0,5,4,0,0,0,0,2,1,\ldots,$$

α=even j, β= odd j; maximum prime tested is 702269.

x	Row n_1	$n_1^/$	$n_2^/$
5	1	{2,3,4,5,8,9,13,14,17,18,22,28, 30,32};{0,1,2,3,5,6,10,11,13}*	{0,2,6,8,11,12,13,15,17,25,26,28, 31,32,33};{3,7,10,12,13}*
11	2	{3,4,6,11,19,26}	{1,2,4,7,11,14,17,22,24,26}
17	3	{1,2,6,7,11,15,16}	{0,3,4,5,10,13,15,19,23,24,25}
23	4	{2,7,12,15,16,17,20,21}; {4,6,7,11,12}*	{3,5,8,9,10,14,20,23,24,25}; {1,3,4,6,13}*
29	5	{1,3,4,6,8,26}	{6,7,9,11,12,14,21,22,26}

Table 6: m=1, $y \in \bar{1}_6$ (row of Q=$n_1^/$) or $y \in \bar{5}_6$ (row of Q=$n_2^/$); *m=2.

4. Concluding Comments

The rows of even squares are often primes, so that, unlike the rows of odd squares, they are not simply defined.

However, the row, R_1, (which contains all even squares when $3 \nmid N$) does have specific REDs which correspond to a particular RED for x. Furthermore, certain REDs for y are incompatible with x* values (Table 7). Of course, when $y \in \bar{3}_6, R_3$, the row of y^2, will have 6 as a factor.

x*	R_1^*	Incompatible y*
1,9	3,7	2,8
3,7	1,7	4,6
5	1,3	0

Table 7: Some examples of incompatible REDs

By extracting twos, we have linked the generalized pentagonal numbers, K_i, for the odd squares with the corresponding K_i values for the odd part of the even squares. (When 3|N, the link is through the triangular numbers.) This gives a unifying picture of the row structure which can be used to advantage when studying systems which feature squares, as illustrated by the examples we have given.

Another system of interest would be Pythagorean Triples which present odd and even square relationships (*cf.* [4]). Indeed, any study which involves squares can be analysed in more depth by using these modular-ring/row structures as a basis for interpretation. For example, consider the primitive Pythagorean triple (233,208,105). The components have the class structure $<\bar{2}_6, \bar{1}_6, \bar{6}_6>$, while their squares have the structure $<\bar{4}_6, \bar{1}_6, \bar{6}_6>$, so that the row function for the squares is

$$R_4 = R_1 + R_6 \tag{4.1}$$

and with $r_6 = 17$,

$$R_6 = 1 + 12\sum_{i=1}^{j} i = 1 + 6\big(17(17+1)\big) = 1837,$$

and

$$R_4 = 4K = 2n_1(3n_1 - 1)$$

and

$$n_1 = r_2 = 39$$

so that

$$R_4 = 9048$$

and

$$R_1 = 2^{2m}\big(6n_2^2 + 2n_2 + (1/6)\big) + (1/3)$$

so that with m=4, Q=13 (row = n_2=2)

$$R_1 = 7211.$$

These results satisfy Equation (4.1).

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