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SOME FERMATIAN SPECIAL FUNCTIONS

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Abstract
Generalizations of the polynomials of Bernoulli, Euler and Hermite are defined here in terms of generalized integers called Fermatian integers. These are closely related to the *q*-series extensively studied by Leonard Carlitz. These various analogues of the classical special functions are inter-related with one another and also to some of the problems posed by Morgan Ward. The works of Henry Gould and Vern Hoggatt are also extensively cited.

1. Introduction

This paper is mainly concerned with generalizations of the Bernoulli, Euler and Hermite polynomials which are based on the use of Fermatian numbers instead of the ordinary integers [25]. Ordinary Bernoulli, Euler and Hermite polynomials can be defined respectively as

$$te^{xt} \Big/ \left(e^t - 1\right) = \sum_{n=0}^{\infty} B_n(x) t^n \Big/ n!, \tag{1.1}$$

$$2e^{xt} \Big/ \left(e^t + 1\right) = \sum_{n=0}^{\infty} E_n(x) t^n \Big/ n!, \tag{1.2}$$

$$e^{xt} \Big/ \left(e^{t^2 - xt}\right) = \sum_{n=0}^{\infty} H_n(x) t^n \Big/ n!. \tag{1.3}$$

Carlitz [1-15] has studied numerous generalizations of these polynomials. In some of them he has developed their q-series analogues [3,6]. Some of the q-Bernoulli numbers and polynomials studied by Carlitz have been:

$$\beta_n(x,q) = (q-1)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{r+1}{\underline{q}_{r+1}} q^{rx}, \tag{1.4}$$

$$\beta_n(q) = \beta_n(0,q) = (q-1)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{r+1}{\underline{q}_{r+1}}, \tag{1.5}$$

in which $\underline{q}_n$ is the *n*-th reduced Fermatian number of index *b*:

$$\underline{q}_n = 1 + q + q^2 + \dots + q^{n-1}, \quad (n > 0) \tag{1.6}$$

with $\underline{q}_0 = 1$, so that

$$\lim_{q \to 1} \beta_n(q) = B_n, \tag{1.7}$$

where B_n is an ordinary Bernoulli number. Since $\beta_n(q)$ is a rational function of *q,* we may put

$$\beta_n(q) = \sum_{r=n}^{\infty} \beta_{n,r}(q-1)^{n-r}, \tag{1.8}$$

so that

$$\beta_{n,n} = B_n.$$

In his studies of various generalizations of Bernoulli numbers, Carlitz usually looked at how they fit in with analogues of the Staudt-Clausen theorem. Horadam and the present writer have also done this [21] and have also studied a relationship between generalized Bernoulli numbers and reciprocals of generalized Fibonacci numbers [26].

It is the purpose of this paper to consider some generalized special functions defined in terms of Fermatian numbers.

2. Fermatian Bernoulli Polynomials

Define

$$\frac{te_z(xt)}{e_z(t)-1} = \sum_{n=0}^{\infty} B_{n,z}(t)\frac{t^n}{\underline{z}_n!} \tag{2.1}$$

where

$$e_z(t) = \sum_{n=0}^{\infty} t^n \Big/ \underline{z}_n!.$$

We shall call $B_{n,z}(x)$ a Fermatian Bernoulli polynomial. Clearly

$$B_{n,1}(x) = B_n(x).$$

$$\begin{aligned}\frac{e_z(t)-1}{t} &= \frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{\underline{z}_n!}\\ &= \sum_{n=1}^{\infty}\frac{t^{n-1}}{\underline{z}_n!}\\ &= \sum_{n=0}^{\infty}\frac{t^n}{\underline{z}_{n+1}!}\end{aligned}$$

This can be applied to the Fermatian Bernoulli polynomial defined above by adapting Carlitz` approach to the ordinary Bernoulli polynomial. Using the above we have that

$$e_z(t) = \sum_{r=0}^{\infty}\frac{t^r}{\underline{z}_{r+1}!}\sum_{n=0}^{\infty} B_{n,z}(x)\frac{t^n}{\underline{z}_n!}.$$

That is,

$$\begin{aligned}\sum_{n=0}^{\infty} x^n\frac{t^n}{\underline{z}_n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{B_{n-k,z}(x)t^n}{\underline{z}_{k+1}\,\underline{z}_k!\,\underline{z}_{n-k}!}\\ &= \sum_{n=0}^{\infty}\frac{t^n}{\underline{z}_n!}\sum_{k=0}^{n}\frac{1}{\underline{z}_{k+1}}\begin{bmatrix} n\\ k\end{bmatrix}B_{n-k,z}(x),\end{aligned}$$

in which

$$\begin{bmatrix} n\\ k\end{bmatrix} = \frac{\underline{z}_n!}{\underline{z}_k!\,\underline{z}_{n-k}!}$$

is the q-series binomial coefficient. Whence we get

$$x^n = \sum_{k=0}^{n} \frac{1}{\underline{z}_{k+1}} \begin{bmatrix} n \\ k \end{bmatrix} B_{n-k,z}(x) \tag{2.2}$$

which is a q-series analogue of the following result which was developed by Carlitz [16]:

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} B_{n-k}(x).$$

3. Fermatian Hermite and Euler Polynomials

Carlitz, in his papers on Hermite polynomials, for example [16], suggested the definition

$$e(tz)e(z) = \sum_{n=0}^{\infty} H_n(t) \, z^n \big/ (x)_n \tag{3.1}$$

in which we have the q-series [6]

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n).$$

It is appropriate at this stage to interrupt with some comments on notation. Since the notation $(q)_n$ is also used in combinatorics for the falling factorial coefficient

$$(q)_n = q(q-1)..(q-n+1),$$

it is worth adopting Knuth's suggestion at the 1967 Conference on Combinatorial Mathematics and its Applications (see Riordan [24]), namely that we write $q^{\underline{n}}$ for the falling factorial coefficient and $q^{\overline{n}}$ for the rising factorial coefficient

$$q^{\overline{n}} = q(q+1)\ldots(q+n-1).$$

The $H_n(t)$ of Equation (3.1) are thus in some ways analogous to the Hermite polynomials defined by

$$e^{2xz-z^2} = \sum_{n=0}^{\infty} H_n(x) \, z^n \big/ n!.$$

We define instead

$$e_z(xt)e(t) = \sum_{n=0}^{\infty} H_{n,z}(x) \, t^n \big/ \underline{z}_n!, \tag{3.2}$$

where the $H_{n,z}(x)$ are Fermatian extensions of the Hermite polynomials. We then have

$$H_{n,z}(x) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{3.3}$$

Proof:

$$\sum_{n=0}^{\infty} H_{n,z}(x) \, t^n \big/ \underline{z}_n! = \sum_{m=0}^{\infty} \frac{x^m t^m}{\underline{z}_m!} \sum_{n=0}^{\infty} \frac{t^n}{\underline{z}_n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k t^n}{\underline{z}_k! \, \underline{z}_{n-k}!}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{\underline{z}_n!} \sum \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

and the result follows when the coefficients of t^n are equated.

Carlitz also suggested that we define an operator $\underset{x}{\Delta}$ by means of

$$\underset{x}{\Delta} f(t) = f(t) - f(xt).$$

If we re-define this for $H_{n,z}(x)$ and use the relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix},$$

we find that

$$\begin{aligned}\underset{x}{\Delta} H_{n,z}(x) &= H_{n,z}(x) - H_{n,z}(zx) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \left(1 - z^{n-k}\right) \\ &= \sum_{k=0}^{n} \frac{\left(1 - z^{n}\right) \underline{z}_{n-1}!}{\underline{z}_{k}!\,\underline{z}_{n-k-1}!} x^{n-k} \\ &= \left(1 - z^{n}\right) x \sum_{k=0}^{n-1} \begin{bmatrix} n-k \\ k \end{bmatrix} x^{n-k-1} \\ &= \left(1 - z^{n}\right) x H_{n-1,z}(x).\end{aligned}$$

This suggests a means of obtaining an analogue of

$$\begin{aligned}\Delta_x B_n(x) &= B_n(x+1) - B_b(x) \\ &= n x^{n-1}\end{aligned}$$

where $B_n(x)$ is an ordinary Bernoulli polynomial. Unfortunately there is no simple expression for $e_z(xt) - e_z(zxt)$ which is necessary to obtain the analogue. The analogues cannot be developed directly because

$$e_z\left((x+1)z\right) \neq e_z(xz) e_z(z)$$

whereas

$$\exp((x+1)z) = \exp(xz)\exp(z).$$

All that can be stated is it that follows from the above that

$$\sum_{k=0}^{n} \frac{1}{\underline{z}_{k+1}} \begin{bmatrix} n \\ k \end{bmatrix} \Delta_x B_{n-k,z}(x) = \sum_{k=0}^{n-1} \binom{n}{k} x^k. \tag{3.4}$$

Proof:

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{\underline{z}_{k+1}} \begin{bmatrix} n \\ k \end{bmatrix} \left\{B_{n-k,z}(x+1) - B_{n-k,z}(x)\right\} &= (x+1)^k - x^k \\ &= \sum_{k=0}^{n-1} \binom{n}{k} x^k.\end{aligned}$$

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Ward [28] bypassed this problem by writing $(x+y)^n$ for the polynomial $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^{n-r} y^r$. Later he examined

$$D : Dx^n = \underline{z}_n x^{n-1}.$$

Ward then defined

$$F^{(r+1)}(x) = DF^{(r)}(x)$$

where

$$F(x) = \sum_{n=0}^{\infty} c_n x^n$$

so that

$$F^{(r)}(x) = D^r F(x).$$

This means that

$$F^{(r)}(x) \Big/ \underline{z}_r! = \sum_{n=r}^{\infty} \begin{bmatrix} n \\ r \end{bmatrix} c_n x^{n-r}.$$

This led Ward to replace

$$F(x+y) = \sum_{n=0}^{\infty} c_n (x+y)^n$$

formally by

$$F(x+y) = \sum_{n=0}^{\infty} F^{(n)}(x)\, y^n \Big/ \underline{z}_n!$$

which is an analogue of Taylor's formula. Thus, in Ward`s notation,

$$e(x+y) = \sum_{n=0}^{\infty} e^{(n)}(x)\, y^n \Big/ \underline{z}_n!$$

$$= e(x)e(y).$$

We do not need Ward's approach to define suitable Bernoulli numbers to which a Staudt-Clausen theorem can be applied, something which Ward was unable to do with his method [21]. Carlitz [3] has another approach, which is mentioned later. Nörlund [23] defined general Bernoulli and Euler polynomials of higher order as follows:

$$\frac{t^t e^{xt}}{(e^t - 1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}$$

so that

$$B_k^{(1)}(x) = B_k(x)$$

and

$$\frac{2^n e^{xt}}{(e^t + 1)^n} = \sum_{k=0}^{\infty} E_k^{(n)}(x) \frac{t^k}{k!}.$$

Generalizations of these have been considered by Horadam and the present writer [26]. The work of Gould [17] should also be note here; he has studied these numbers at length and has proved such elegant formulas as

$$B_k^{(z)} = \sum_{j=0}^{n} (-1)^j \binom{k+1}{j+1} B_k^{(-jz)}.$$

Leopoldt [22] has defined another generalized Bernoulli number B_ξ^n where ξ denotes a primitive character (mod f):

$$\sum_{r=i}^{p}\xi(r)\frac{te^{(r-\xi)t}}{e^{pt}-1}=\sum_{n=1}^{\infty}B_{\xi}^{n}(x)\frac{t^n}{n!}$$

so that

$$\begin{aligned}B_{\xi}^{n} &= p^{n-1}\sum_{r=1}^{p-1}\xi(r)B_n\frac{r}{p}\\ &= B_{\xi}^{n}(0);\end{aligned}$$

when p=1, we get the ordinary Bernoulli numbers. Carlitz [10] refined some of Leopoldt's results: let

$$\frac{t^n e_z(xt)}{\left(e_z(t)-1\right)^n}=\sum_{k=0}^{\infty}B_{k,z}^{(n)}\frac{t^k}{\underline{z}_k!}. \qquad (3.5)$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty}B_{k,z}(x)\frac{t^n}{\underline{z}_n!}&=\frac{te_z(xt)}{e_z(t)-1}\\ &=\frac{te_z(xt)e_z(t)}{\left(e_z(t)-1\right)^2}-\frac{t^2e_z(t)}{t\left(e_z(t)-1\right)^2}\\ &=\frac{t}{\left(e_z(t)-1\right)^2}\sum_{k=0}^{\infty}H_{k,z}(x)\frac{t^k}{\underline{z}_k!}-\frac{1}{t}\sum_{k=0}^{\infty}B_{k,z}^{(2)}(x)\frac{t^k}{\underline{z}_k!},\end{aligned}$$

which relates the analogues of the Hermite polynomials to the analogues of the Bernoulli polynomials. This relationship can be made more specific:

$$\begin{aligned}\left(\frac{e_z(t)-1}{t}\right)^2&=\sum_{n=0}^{\infty}\frac{t^n}{\underline{z}_{n+1}!}\sum_{m=0}^{\infty}\frac{t^m}{\underline{z}_{m+1}!}\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^n}{\underline{z}_{n-r+1}!\,\underline{z}_{r+1}!}\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix}n+2\\ r+1\end{bmatrix}\frac{t^n}{\underline{z}_{n+2}!}.\end{aligned}$$

Let us formally define $B_{-1,z}(x)/\underline{z}_{-1}!$ to be zero. Then

$$\begin{aligned}\left(\frac{e_z(t)-1}{t}\right)^2\sum_{k=0}^{\infty}B_{k,z}(x)\frac{t^{k+1}}{\underline{z}_k!}&=\left(\frac{e_z(t)-1}{t}\right)^2\sum_{k=0}^{\infty}B_{k-1,z}(x)\frac{t^k}{\underline{z}_{k-1}!}\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\begin{bmatrix}n+2\\ r+1\end{bmatrix}\frac{t^n}{\underline{z}_{n+2}!}\sum_{k=0}^{\infty}B_{k-1,z}(x)\frac{t^k}{\underline{z}_{k-1}!}\\ &=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{r=0}^{n}\begin{bmatrix}k+1\\ n+2\end{bmatrix}\begin{bmatrix}n+2\\ r+1\end{bmatrix}B_{k-n-1,z}(x)\frac{t^k}{\underline{z}_{k+1}!}.\end{aligned}$$

Similarly,

$$\left(\frac{e_z(t)-1}{t}\right)^2\sum_{k=0}^{\infty}B_{k,z}^{(2)}(x)\frac{t^k}{\underline{z}_k!}=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{r=0}^{n}\begin{bmatrix}k+2\\ n+2\end{bmatrix}\begin{bmatrix}n+2\\ r+1\end{bmatrix}B_{k-n,z}^{(2)}(x)\frac{t^k}{\underline{z}_{k+2}!}.$$

We can then obtain for the Fermatian Hermite polynomials that

$$H_{k,z}(x) = \sum_{n=0}^{k}\sum_{r=0}^{n}\begin{bmatrix}k+1\\n+2\end{bmatrix}\begin{bmatrix}n+2\\r+1\end{bmatrix}\frac{B_{k-n-1,z}(x)+B^{(2)}_{k-n,z}(x)/\underline{z}_{k-n}}{\underline{z}_{k+1}}. \tag{3.6}$$

For the ordinary Bernoulli polynomials of orders 1 and 2, and for a Hermite polynomial defined by

$$\exp((1+x)t) = \sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!},$$

we have that

$$H'_k(x) = \sum_{n=0}^{k}\sum_{r=0}^{n}\binom{k+1}{n+2}\binom{n+2}{r+1}\frac{B_{k-n-1}(x)+B^{(2)}_{k-n}(x)/(k-n)}{k+1}. \tag{3.7}$$

4. Ordinary Euler and Hermite Polynomials

This suggests that we try to discover a similar relationship among ordinary Euler polynomials and Hermite polynomials defined by

$$\exp((2x-t)t) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

From the ordinary Euler polynomial defined in (1.2) we can obtain

$$\begin{aligned}(e^t+1)\sum_{n=0}^{\infty} E_n(2x)\frac{t^n}{n!} &= 2e^{t^2}\left(e^{(2x-t)t}\right)\\ &= 2e^{t^2}\sum_{n=0}^{\infty} H_n\frac{t^n}{n!}.\end{aligned}$$

Now,

$$\begin{aligned}(e^t+1)\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty}\frac{t^m}{m!}+1\right)\sum_{n=0}^{\infty} E_n(2x)\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty} E_n(2x)\frac{t^n}{n!}+\sum_{n=0}^{\infty} E_n(2x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r}E_r(2x)+E_n(2x)\right\}\frac{t^n}{n!},\end{aligned}$$

and

$$\begin{aligned}2e^{t^2}\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} &= 2\sum_{m=0}^{\infty}\frac{t^{2m}}{m!}\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\\ &= 2\sum_{n=0}^{\infty}\sum_{r=\lfloor n/2\rfloor}^{n} H_{2r-n}(x)\frac{t^n}{(n-r)!(2r-n)!}\\ &= 2\sum_{n=0}^{\infty}\sum_{r=\lfloor n/2\rfloor}^{\infty}\frac{r!}{(2r-n)!}\binom{n}{r}H_{2r-n}(x)\frac{t^n}{n!}.\end{aligned}$$

On equating coefficients of t^n, we get

$$E_n(x) + \sum_{r=0}^{n} \binom{n}{r} E_r(2x) = 2 \sum_{r=\lfloor n/2 \rfloor}^{n} \frac{r!}{(2r-n)!} \binom{n}{r} H_{2r-n}(x), \tag{4.1}$$

which is a relation between the Euler and Hermite polynomials.

5. Conclusion

It is of interest in closing to consider generalized Bernoulli and Euler polynomials analogous to those of Gould [18]. Let

$$\frac{tC(tx)}{C(t)-1} = \sum_{k=0}^{\infty} B_{k,z}(x,c) \frac{t^k}{\underline{z}_k!} \tag{5.1}$$

and

$$\frac{2C(tx)}{C(t)+1} = \sum_{k=0}^{\infty} E_{k,z}(x,c) \frac{t^k}{\underline{z}_k!} \tag{5.2}$$

define $B_{k,z}(x,c)$ and $E_{k,z}(x,c)$, in which

$$C(t) = e_z(ct)$$

where $e_z(t)$ is the Fermatian exponential. This is analogous to the ordinary situation where

$$C^t = e^{ct}, \text{ if } C = e^c.$$

In Gould's work, $C = \beta/\alpha$, where $\alpha = \frac{1}{2}\left(1+\sqrt{5}\right)$ and $\beta = \frac{1}{2}\left(1-\sqrt{5}\right)$ are the roots of the Fibonacci characteristic polynomial $x^2 - x - 1 = 0$. Incidentally, Gould's C and Hoggatt's C_{nk} [19] can be related when $p = -q = 1$ in the characteristic polynomial of Horadam's generalized Fibonacci numbers [20] $x^2 - px + q = 0$:

$$C = \beta \lim_{k\to\infty} \left(C_{k+1,k+1} \big/ C_{k,k} \right). \tag{5.3}$$

Proof:

$$C_{k,k} = \frac{F_{k-1}F_{k-2}\ldots F_1}{F_1F_2\ldots F_{k-1}F_k} \quad \text{when } p = -q = 1$$
$$= 1 \big/ F_k,$$

where F_k is the kth Fibonacci number..

$$\beta \lim_{k\to\infty} \frac{C_{k+1,k+1}}{C_{k,k}} = \beta \lim_{k\to\infty} \frac{F_k}{F_{k+1}}$$
$$= \frac{\beta}{\alpha}$$

from Vorob`ev [27]. From (5.1) we get

$$\sum_{k=0}^{\infty} B_{k,z}(x,c)\frac{t^k}{k!} = \frac{te_z(ctx)}{e_z(ct)-1}$$
$$= \frac{1}{c}\frac{cte_z(ctx)}{e_z(ct)-1}$$
$$= \frac{1}{c}\sum_{k=0}^{\infty} B_{k,z}(x)\frac{(ct)^k}{\underline{z}_k!}$$

which gives

$$B_{k,z}(x,c) = B_{k,z}(x)c^{k-1} \tag{5.4}$$

as a relation between the analogues of the ordinary and Fermatian Bernoulli polynomials. A similar relation can be found for Euler polynomials. When z=1 we get the corresponding relation for ordinary Bernoulli polynomials

$$B_b(x,c) = B_k(x)(\log C)^{k-1},$$

which agrees with Gould.

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