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# ITERATED DIRICHLET SERIES AND THE INVERSE OF RAMANUJAN'S SUM

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The theory of Dirichlet series having number theoretic functions of a single variable as coefficients has a rich history. In this paper we present a parallel theory for iterated Dirichlet series with number theoretic functions of two variables as coefficients and find the Dirichlet product inverse of Ramanujan's sum. The results presented here are easily accessible to an Advanced Cakculus or undergraduate Number Theory course.

Adopting the traditional notation, for the complex variable s we write $s = \sigma + it$. Ramanujan [3] introduced the function C(n;k), the sum of the $n^{th}$ powers of the $k^{th}$ primitive roots of unity and showed that:

$$C(n;k) = \sum_{d|(k,n)} d\,\mu(k/d),$$

where $\mu$ is the Mobius function. A function f, of one variable, is said to be multiplicative if f(1) = 1 and f(mn) = f(m)f(n) whenever (m,n) = 1. A function f(n;k) of two variables is said to be multiplicative if f(1;1) = 1 and $f(n_1n_2;k_1k_2) = f(n_1;k_1)\, f(n_2;k_2)$ whenever $(n_1k_1, n_2k_2) = 1$. We denote by $*$ the usual Dirichlet product of one variable, i.e.

$$f * g(n) = \sum_{d|n} f(d)\, g(n/d),$$

where d|n denotes that the sum is taken over the positive divisors of n. The usual Dirichlet product for functions of two variables is denoted by $**$, thus

$$f ** g(n;k) = \sum_{d_1|n} \sum_{d_2|k} f(d_1;d_2)\, g(n/d_1; k/d_2).$$

The function $\delta(n) = 1$ if n = 1 and 0 otherwise is the identity for the $*$ product, while $\delta(n;k) = \delta(n)\delta(k)$ is the identity for the $**$ product.

We know that a Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges in some half-plane $\sigma > \sigma_c$ and is analytic there with derivative

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n) \log n}{n^s}.$$

Each Dirichlet series also converges absolutely on some half-plane $\sigma > \sigma_a$ with $0 \leq \sigma_a - \sigma_c \leq 1$. If another such function $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ shares a common half-plane of absolute convergence with F, then in that half-plane we have:

$$F(s)\,G(s) = \sum_{n=1}^{\infty} \frac{f*g(n)}{n^s} \quad [2].$$

**1. Definition:** By an iterated Dirichlet series we mean a series of the form:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{F(n;k)}{(nk)^s}.$$

Following the proofs in [2], making adjustments for the iterated series, it is straightforward to prove that such functions converge on some half-plane $\sigma > \sigma_c$ and converge absolutely on some half-plane $\sigma > \sigma_a$ where $-\infty \leq \sigma_c \leq \sigma_a \leq \infty$ and $0 \leq \sigma_a - \sigma_c \leq 1$. The proofs of these facts are suitable exercises for an Advanced Calculus class. As in the case of single Dirichlet series we have that iterated Dirichlet series, F(s), are analytic functions of s in their half-plane of convergence with derivative

$$F'(s) = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{f(n;k) \log nk}{(nk)^s}.$$

**2. Theorem:** Let $F(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{A(n;k)}{(nk)^s}$ and $G(s) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{B(i;j)}{(ij)^s}$. Then in the common half-plane of absolute convergence we have:

$$F(s)G(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{A{*}{*}B(n;k)}{(nk)^s}.$$

Proof:

$$F(s)G(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{A(n;k)}{(nk)^s} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{B(i;j)}{(ij)^s}$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{A(n;k)B(i;j)}{(nkij)^s}.$$

Since the convergence is absolute, we may collect all of the terms for which n is a constant, say m, and those for which kj is another constant, say r. The possible values for m and r are 1, 2, 3, ... so we may rewrite the last expression as

$$= \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sum_{ni=m} \sum_{kj=r} \frac{A(n;k)B(i;j)}{(mr)^s}$$

$$= \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sum_{n|m} \sum_{k|r} \frac{A(n;k)B(m/n\,;\,r/k)}{(mr)^s}$$

$$= \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{A{*}{*}B(m;r)}{(mr)^s}.$$

Ramanujan [2] showed that for $\sigma > 0$:

$$\sum_{k=1}^{\infty} \frac{C(n;k)}{k^{s+1}} = \frac{\sigma_s(n)}{n^s\,\zeta(s+1)},$$

where $\sigma_s$ is the sum of the $s^{th}$ powers of the divisors of n and $\zeta$ is the Riemann zeta function. Using the result of Ramanujan and Euler products we can evaluate an iterated Dirichlet series. Recall that if f is a multiplicative function with $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ absolutely convergent, then we may write the sum as an infinite product of the form:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} \cdots\right).$$

where $\prod_p$ denotes that the product is taken over all primes p.

**3. Lemma:** For $\sigma > 1$ we have:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C(n;k)}{(nk)^s} = \zeta(2s-1)\,.$$

Proof: The given sum converges absolutely for $\sigma > 1$ [3].

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{C(n;k)}{(nk)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{\infty} \frac{C(n;k)}{k^s}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}, \text{ from the result of Ramanujan,}$$

$$= \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\sigma_{s-1}(n)}{n^{2s-1}}.$$ The summand is multiplicative in n

and we may expand it in an Euler product so that the last equality becomes:

$$= \frac{1}{\zeta(s)} \prod_p \left( 1 + \frac{\sigma_{s-1}(p)}{p^{2s-1}} + \frac{\sigma_{s-1}(p^2)}{p^{2(2s-1)}} + \cdots \right)$$

$$= \frac{1}{\zeta(s)} \prod_p \left( 1 + \frac{1+p^{s-1}}{p^{2s-1}} + \frac{1+p^{s-1}+p^{2(s-1)}}{p^{2(2s-1)}} + \cdots \right)$$

$$= \frac{1}{\zeta(s)} \prod_p \left( 1 + \frac{p^{s-1}}{p^{2s-1}} + \frac{p^{2(s-1)}}{p^{2(2s-1)}} + \cdots \right)\left( 1 + \frac{1}{p^{2s-1}} + \frac{1}{p^{2(2s-1)}} + \cdots \right)$$

$$= \frac{1}{\zeta(s)} \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right)\left( 1 + \frac{1}{p^{2s-1}} + \frac{1}{p^{2(2s-1)}} + \cdots \right)$$

$$= \frac{1}{\zeta(s)} \zeta(s)\,\zeta(2s-1)$$

$$= \zeta(2s-1)$$

Anderson and Apostol [1] define a generalized Ramanujan sum to be a function of the form $S(n;k) = \sum_{d|(k,n)} f(d)\, g(k/d)$. It is now useful for us to extend this definition slightly and include a function of the variable k in the definition.

**4. Definition**. By a generalized Ramanujan sum, we mean a function of the form:

$$S(n;k) = \sum_{d|(k,n)} f(d)\, g(k/d)\, h(n/d).$$

Now consider the function of two variables $A(n;k) = \sum_{d|(n,k)} d\mu(d)\, \mu(n/d)$. It is simple to show that the function A(n;k) is multiplicative in two variables and is also multiplicative in the variable n for fixed k. The values for A on powers of the same prime are given by:

$$A(p^a; p^b) = \mu(p^a) \text{ if } b = 0,$$

and otherwise by:

$$A(p^a; p^b) = \begin{cases} -(p+1) & \text{if } a = 1 \\ p & \text{if } a = 2 \\ 0 & a > 2. \end{cases}$$

Using this information we can now evaluate a traditional Dirichlet series:

**5. Lemma:** For $\sigma > 1$:

$$\sum_{n=1}^{\infty} \frac{A(n;k)}{n^s} = \frac{\phi_{s-1}(k)}{\zeta(s)\, k^{s-1}}, \quad \text{where } \phi_s(k) = k^s \prod_{p|k}(1 - \tfrac{1}{p^s}).$$

Proof: The given sum converges absolutely for $\sigma > 1$ and since the summand is a multiplicative function of n we may expand it in an Euler product. Thus for fixed k,

$$\sum_{n=1}^{\infty} \frac{A(n;k)}{n^s} = \prod_{p} \left(1 + \frac{A(p;k)}{p^s} + \frac{A(p^2;k)}{p^{2s}} + \cdots\right)$$

$$= \prod_{p|k} \left(1 + \frac{-(p+1)}{p^s} + \frac{p}{p^{2s}}\right) \prod_{p\nmid k}\left(1 + \frac{-1}{p^s}\right)$$

$$= \prod_{p|k} \frac{\left(1 + \frac{-(p+1)}{p^s} + \frac{p}{p^{2s}}\right)}{\left(1 - \frac{1}{p^s}\right)} \prod_{p}\left(1 - \frac{1}{p^s}\right)$$

$$= \frac{1}{\zeta(s)} \prod_{p|k}\left(1 - \frac{1}{p^{s-1}}\right)$$

$$= \frac{1}{k^{s-1}\zeta(s)} k^{s-1} \prod_{p|k} \left(1 - \frac{1}{p^{s-1}}\right)$$

$$= \frac{\phi_{s-1}(k)}{\zeta(s)\, k^{s-1}}.$$

With this intermediate result we can now evaluate another iterated Dirichlet series:

**6. Lemma:** For $\sigma > 1$,

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{A(n;k)}{(nk)^s} = \frac{1}{\zeta(2s-1)}.$$

Proof: For $\sigma > 1$,

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{A(n;k)}{(nk)^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} \frac{\phi_{s-1}(k)}{\zeta(s)\, k^{s-1}}, \text{ by the previous lemma,}$$

$$= \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{\phi_{s-1}(k)}{k^{2s-1}}.$$ Expanding in an Euler product this becomes:

$$= \frac{1}{\zeta(s)} \prod_{p} \left(1 + \frac{\phi_{s-1}(p)}{p^{2s-1}} + \frac{\phi_{s-1}(p^2)}{p^{2(2s-1)}} + \cdots\right)$$

$$= \frac{1}{\zeta(s)} \prod_{p} \left(1 + \frac{p^{s-1}-1}{p^{2s-1}} + \frac{p^{2(s-1)} - p^{(s-1)}}{p^{2(2s-1)}} + \frac{p^{3(s-1)} - p^{2(s-1)}}{p^{3(2s-1)}} + \cdots\right)$$

$$= \frac{1}{\zeta(s)} \prod_{p} \left(1 + \frac{p^{s-1}}{p^{2s-1}} + \frac{p^{2(s-1)}}{p^{2(2s-1)}} + \frac{p^{3(s-1)}}{p^{3(2s-1)}} + \cdots\right)\left(1 - \frac{1}{p^{(2s-1)}}\right)$$

$$= \frac{1}{\zeta(s)} \prod_{p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)\left(1 - \frac{1}{p^{(2s-1)}}\right)$$

$$= \frac{1}{\zeta(2s-1)}.$$

**7. Theorem:** The function A defined above is the Dirichlet inverse of Ramanujan's sum.

Proof: The product of the iterated Dirichlet series in Lemmas 3 and 6 is $1 = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\delta(n;k)}{(nk)^s}$ .

Thus by Theorem 2, $C(n;k) * A(n;k) = \delta(n;k)$ and the function A given above is the Dirichlet inverse of Ramanujan's sum, $C^{-1}(n;k)$.

It is not difficult to show that D(n;k), the number of divisors of k which are relatively prime to n, is a generalized Ramanujan sum given by $D(n;k) = \sum_{d|(k,n)} \mu(d)\, \tau(k/d)$ , where $\tau$ is the number of divisors of n. The function S(n;k), the sum of the divisors of k which are relatively prime to n, is given by $S(n;k) = \sum_{d|(k,n)} d\mu(d)\, \sigma(k/d)$ , where $\sigma$ is the sum of the divisors of n. Using the methods above, the following evaluations of Dirichlet series make appropriate exercises:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{D(n;k)}{(nk)^s} = \frac{1}{\zeta(2s)} \zeta^3(s) \text{ and}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{D(n;k)}{(nk)^s} = \frac{1}{\zeta(2s-1)} \zeta^2(s)\, \zeta(s-1).$$

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