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Properties of the restrictive factor

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In [1] Krassimir Atanassov has introduced the arithmetic function $RF(n)$ defined by $RF(1) = 1$ and $RF(n) = q_1^{\alpha_1-1}q_2^{\alpha_2-1}\cdots q_k^{\alpha_k-1}$ whenever $n = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$, where $q_1,\dots,q_k$ are pairwise different prime numbers and we have $\alpha_1,\dots,\alpha_k \geq 1$. The function $RF(n)$ is called the *restrictive factor.* In the present paper we denote it simply by $R(n)$.

Denoting as usual Euler's function by $\varphi(n)$ and the sum of all divisors of n by $\sigma(n)$, one proved in [1] that

$$n^2 - \varphi(n)\sigma(n) \geq n^2. \tag{1}$$

In connection with this inequality we now prove the following fact.

Theorem 1 *For $n = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$ and $\alpha \geq 1/k$ we have*

$$R^\alpha(n) + \varphi^\alpha(n)\sigma^\alpha(n) \leq n^{2\alpha}. \tag{2}$$

Proof. We have $\varphi(n) = n(1-1/q_1)(1-1/q_2)\cdots(1-1/q_k)$, $\sigma(n) = (q_1^{\alpha_1+1} - 1)(q_2^{\alpha_2+1}-1)\cdots(q_k^{\alpha_k+1}-1)/((q_1-1)(q_2-1)\cdots(q_k-1))$ and $R(n) = n/(q_1q_2\cdots q_k)$. Thus inequality (2) takes the form

$$\frac{1}{\left(q_1^{\alpha_1+1}q_2^{\alpha_2+1}\cdots q_k^{\alpha_k+1}\right)^\alpha} + \left(\frac{q_1^{\alpha_1+1}-1}{q_1-1}\cdot\frac{q_2^{\alpha_2+1}-1}{q_2-1}\cdots\frac{q_k^{\alpha_k+1}-1}{q_k-1}\right)^\alpha \leq 1.$$

Consequently, denoting $1/q_i^{\alpha_i+1} = x_i$ for $i = 1, 2, \dots, k$, we have to prove that

$$(x_1x_2\cdots x_k)^\alpha + \left((1-x_1)(1-x_2)\cdots(1-x_k)\right)^\alpha \leq 1. \tag{3}$$

Since $x_i \in (0,1)$, $1 - x_i \in (0,1)$ and $\alpha \geq 1/k$, we have $(x_1x_2\cdots x_k)^\alpha \leq (x_1x_2\cdots x_k)^{1/k}$ and $\left((1-x_1)(1-x_2)\cdots(1-x_k)\right)^\alpha \leq \left((1-x_1)(1-x_2)\cdots(1-x_k)\right)^{1/k}$. According to the inequality between the arithmetic mean and the geometric one we have $(x_1x_2\cdots x_k)^{1/k} \leq (x_1+x_2+\cdots+x_k)/k$ and $\left((1-x_1)(1-x_2)\cdots(1-x_k)\right)^{1/k} \leq ((1-x_1)+(1-x_2)+\cdots+(1-x_k))/k$, whence

$$\begin{aligned}(x_1x_2\cdots x_k)^\alpha + &\left((1-x_1)(1-x_2)\cdots(1-x_k)\right)^\alpha \\ &\leq \frac{x_1+x_2+\cdots+x_k}{k} + \frac{k - x_1 - x_2 - \cdots - x_k}{k} \\ &= 1,\end{aligned}$$

and the proof ends. ■

In the following we shall study the arithmetic mean sequence, as well as the geometric one, for the function $R(n)$. To this end we denote $G_n = \sqrt[n]{R(1)R(2)\cdots R(n)}$ and $A_n = \big(R(1) + R(2) + \cdots + R(n)\big)/n$.

Theorem 2 *We have*

$$\lim_{n\to\infty} G_n = e^a \text{ and } \lim_{n\to\infty} A_n = \infty,$$

where $a = \sum_p \log p/(p(p-1))$.

To prove this theorem, we need two lemmas.

Lemma 1 *If* $\gamma(n)$ *stands for the core of* n*, that is, the function defined by* $\gamma(1) = 1$ *and* $\gamma(n) = \prod_{p|n} p$*, then we have*

$$\sum_{n\le x} \log \gamma(n) = x\log x - (a+1)x + O(\sqrt{x}), \tag{4}$$

where $a = \sum_p \log p/(p(p-1))$.

The proof of this fact can be found in [2].

Next, denoting $S(n) = \sum_{k=1}^{n} R(k)$, we have the following inequality.

Lemma 2 *We have*

$$S(n) \ge \sum_{p\le n} p\left[\frac{n}{p^2}\right] \tag{5}$$

whenever $n \ge 1$.

Proof. We are going to prove the desired conclusion by induction. It can be easily checked for $n = 1$ and $n = 2$.

Next note that $S(n) = S(n-1) + R(n) \ge \sum_{p\le n-1} p[(n-1)/p^2] + n/\gamma(n)$. If n is square free, then $[(n-1)/p^2] = [n/p^2]$ and we get $S(n) \ge \sum_{p\le n-1} p[n/p^2] + n/\gamma(n) = \sum_{p\le n-1} p[n/p^2] + 1 > \sum_{p\le n} p[n/p^2]$. The latter inequality follows since $[n/n^2] = 0$ if n is prime.

If $q_1 < q_2 < \cdots < q_i$ are the prime factors of n with $q_1^2 \mid n$, $q_2^2 \mid n$, ... , $q_i^2 \mid n$, then we have $[n/q_j^2] = [(n-1)/q_j^2] + 1$ for $j = 1, 2, \ldots, i$. Consequently

$$\begin{aligned} S(n) &= S(n-1) + R(n) \\ &\ge \sum_{p\le n-1} p\left[\frac{n-1}{p^2}\right] + q_1q_2\ldots q_i \\ &\ge \sum_{p\le n} p\left[\frac{n}{p^2}\right] - (q_1 + q_2 + \cdots + q_i) + q_1q_2\ldots q_i \\ &\ge \sum_{p\le n} p\left[\frac{n}{p^2}\right], \end{aligned}$$

since $q_1q_2\ldots q_i \ge q_1 + q_2 + \cdots + q_i$. The latter inequality can be easily proved by induction since $q_j \ge 2$ for $j = 1, 2, \ldots, i$. ■

Proof of Theorem 2. We have

$$\log G_n = \frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{\gamma(k)} = \frac{1}{n}\left(\log n! - \sum_{k=1}^{n}\log\gamma(k)\right).$$

Stirling's formula shows that $\log n! = n\log n - n + O(\log n)$ and then Lemma 1 implies that

$$\log G_n = \frac{1}{n}\left(n\log n - n + O(\log n) - n\log n + (a+1)n + O(\sqrt{n})\right),$$

that is,

$$\log G_n = a + O\Big(\frac{1}{\sqrt{n}}\Big),$$

whence $\lim\limits_{n\to\infty} G_n = \mathrm{e}^a$ with $a = \sum\limits_p \log p/(p(p-1))$.

Next, it follows by Lemma 2 that

$$\begin{aligned} A_n = \frac{S(n)}{n} &\geq \frac{1}{n}\sum_{p\leq n} p\left[\frac{n}{p^2}\right] = \frac{1}{n}\sum_{p\leq\sqrt{n}} p\left[\frac{n}{p^2}\right] \\ &> \frac{1}{n}\sum_{p\leq\sqrt{n}} p\left(\frac{n}{p^2}-1\right) = \sum_{p\leq\sqrt{n}}\frac{1}{p} - \frac{1}{n}\sum_{p\leq\sqrt{n}} p \\ &> \sum_{p\leq\sqrt{n}}\frac{1}{p} - \frac{1}{n}\sum_{k=1}^{[\sqrt{n}]} k = \sum_{p\leq\sqrt{n}}\frac{1}{p} - \frac{[\sqrt{n}]([\sqrt{n}]+1)}{2n}. \end{aligned}$$

Since the series $\sum\limits_p 1/p$ is divergent and $\lim\limits_{n\to\infty}[\sqrt{n}]([\sqrt{n}]+1)^p/(2n) = 1/2$, it follows that $\lim\limits_{n\to\infty} A_n = \infty$, as asserted. ■

References

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