

On Additive Analogues of Certain Arithmetic Functions

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1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

$$S(n) = \min\{m \in \mathbb{N} : n|m!\}, \quad (1)$$

$$Z(n) = \min\left\{m \in \mathbb{N} : n \mid \frac{m(m+1)}{2}\right\}, \quad (2)$$

$$S_p(n) = \min\{m \in \mathbb{N} : p^n|m!\} \text{ for fixed primes } p. \quad (3)$$

The duals of S and Z have been studied e.g. in [2], [5], [6]:

$$S_*(n) = \max\{m \in \mathbb{N} : m!|n\}, \quad (4)$$

$$Z_*(n) = \max\left\{m \in \mathbb{N} : \frac{m(m+1)}{2}|n\right\}. \quad (5)$$

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

$$S_{p*}(n) = \max\{m \in \mathbb{N} : m!|p^n\} \quad (6)$$

This dual will be studied in a separate paper (in preparation).

2. The additive analogues of the functions S and S_* are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler's gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler's gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of S and S_* from (1) and (4) have been introduced in [3] as follows:

$$S(x) = \min\{m \in \mathbb{N} : x \leq m!\}, \quad S : (1, \infty) \rightarrow \mathbb{R}, \quad (7)$$

resp.

$$S_*(x) = \max\{m \in \mathbb{N} : m! \leq x\}, \quad S_* : [1, \infty) \rightarrow \mathbb{R} \quad (8)$$

Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log \log x} \quad (x \rightarrow \infty) \quad (9)$$

(the same for $S(x)$).

Theorem 2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha} \quad (10)$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (the same for $S_(n)$ replaced by $S(n)$).*

3. The additive analogues of Z and Z_* from (2), resp. (4) will be defined as

$$Z(x) = \min \left\{ m \in \mathbb{N} : x \leq \frac{m(m+1)}{2} \right\}, \quad (11)$$

$$Z_*(x) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \leq x \right\} \quad (12)$$

In (11) we will assume $x \in (0, +\infty)$, while in (12) $x \in [1, +\infty)$.

The two additive variants of $S_p(n)$ of (3) will be defined as

$$P(x) = S_p(x) = \min\{m \in \mathbb{N} : p^x \leq m!\}; \quad (13)$$

(where in this case $p > 1$ is an arbitrary fixed real number)

$$P_*(x) = S_{p*}(x) = \max\{m \in \mathbb{N} : m! \leq p^x\} \quad (14)$$

From the definitions follow at once that

$$Z(x) = k \Leftrightarrow x \in \left(\frac{(k-1)k}{2}, \frac{k(k+1)}{2} \right] \text{ for } k \geq 1 \quad (15)$$

$$Z_*(x) = k \Leftrightarrow x \in \left[\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2} \right) \quad (16)$$

For $x \geq 1$ it is immediate that

$$Z_*(x) + 1 \geq Z(x) \geq Z_*(x) \quad (17)$$

Therefore, it is sufficient to study the function $Z_*(x)$.

The following theorems are easy consequences of the given definitions:

Theorem 3.

$$Z_*(x) \sim \frac{1}{2}\sqrt{8x+1} \quad (x \rightarrow \infty) \quad (18)$$

Theorem 4.

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha} \text{ is convergent for } \alpha > 2 \quad (19)$$

and divergent for $\alpha \leq 2$. The series $\sum_{n=1}^{\infty} \frac{1}{n(Z_(n))^\alpha}$ is convergent for all $\alpha > 0$.*

Proof. By (16) one can write $\frac{k(k+1)}{2} \leq x < \frac{(k+1)(k+2)}{2}$, so $k^2 + k - 2x \leq 0$ and $k^2 + 3k + 2 - 2x > 0$. Since the solutions of these quadratic equations are $k_{1,2} = \frac{-1 \pm \sqrt{8x+1}}{2}$, resp. $k_{3,4} = \frac{-3 \pm \sqrt{8x+1}}{2}$, and remarking that $\frac{\sqrt{8x+1}-3}{2} \geq 1 \Leftrightarrow x \geq 3$, we obtain that the solution of the above system of inequalities is:

$$\begin{cases} k \in \left[1, \frac{\sqrt{1+8x}-1}{2}\right] & \text{if } x \in [1, 3); \\ k \in \left(\frac{\sqrt{1+8x}-3}{2}, \frac{\sqrt{1+8x}-1}{2}\right] & \text{if } x \in [3, +\infty) \end{cases} \quad (20)$$

So, for $x \geq 3$

$$\frac{\sqrt{1+8x}-3}{2} < Z_*(x) \leq \frac{\sqrt{1+8x}-1}{2} \quad (21)$$

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^{\theta}}$ is convergent only for $\theta > 1$.

The things are slightly more complicated in the case of functions P and P_* . Here it is sufficient to consider P_* , too.

First remark that

$$P_*(x) = m \Leftrightarrow x \in \left[\frac{\log m!}{\log p}, \frac{\log(m+1)!}{\log p}\right). \quad (22)$$

The following asymptotic results have been proved in [3] (Lemma 2) (see also [6], p. 172)

$$\log m! \sim m \log m, \quad \frac{m \log \log m!}{\log m!} \sim 1, \quad \frac{\log \log m!}{\log \log(m+1)!} \sim 1 \quad (m \rightarrow \infty) \quad (23)$$

By (22) one can write

$$\frac{m \log \log m!}{\log m!} - \frac{m}{\log m!} \log \log p \leq \frac{m \log x}{\log m!} \leq \frac{m \log \log(m+1)!}{\log m!} - (\log \log p) \frac{m}{\log m!},$$

giving $\frac{m \log x}{\log m!} \rightarrow 1$ ($m \rightarrow \infty$), and by (23) one gets $\log x \sim \log m$. This means that:

Theorem 5.

$$\log P_*(x) \sim \log x \quad (x \rightarrow \infty) \quad (24)$$

The following theorem is a consequence of (24), and a convergence theorem established in [3]:

Theorem 6. *The series $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\log \log n}{\log P_*(n)}\right)^{\alpha}$ is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.*

Indeed, by (24) it is sufficient to study the series $\sum_{n \geq n_0}^{\infty} \frac{1}{n} \left(\frac{\log \log n}{\log n}\right)^{\alpha}$ (where $n_0 \in \mathbb{N}$

is a fixed positive integer). This series has been proved to be convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$ (see [6], p. 174).

References

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