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REMARK ON $n!$

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In D. Mitrinović and M. Popadić's book [1] the inequality

$$n^{\frac{n}{2}} < n! < (\frac{n}{2})^n \tag{1}$$

is introduced, where natural number $n \geq 6$. Here we shall generalize (1), proving
THEOREM: For all natural numbers $n \geq 11$

$$2n^{\frac{n+3}{2}} < n! < \frac{n^{n-\frac{1}{2}}}{2^n}. \tag{2}$$

Proof: First, we note that for $n \geq 9$:

$$(1+\frac{1}{n})^{n+2} < e.(1+\frac{1}{n})^2 < 3e < n$$

(this inequality is valid for smaller n, too). Therefore $n^{n+3} > (n+1)^{n+2}$, i.e.,

$$n^{\frac{n+3}{2}} > (n+1)^{\frac{n+2}{2}}$$

and hence

$$(n+1).n^{\frac{n+3}{2}} > (n+1)^{\frac{n+4}{2}}. \tag{3}$$

Second, we see that for $n \geq 11$

$$(\frac{n+1}{n})^n > (\frac{12}{11})^{11} > 2.604... > 2.088... > 2\sqrt{\frac{12}{11}} > 2\sqrt{\frac{n+1}{n}},$$

i.e.

$$(\frac{n+1}{n})^{n-\frac{1}{2}} > 2. \tag{4}$$

Now, we check directly that (2) is valid for $n = 11$. Let us assume that (2) is valid for some natural number $n \geq 11$. Then we obtain from (3) and (4) sequentially:

$$2(n+1)^{\frac{n+4}{2}} < 2(n+1).n^{\frac{n+3}{2}} < (n+1).n!$$

$$= (n+1)!$$

$$= (n+1).n! \leq (n+1).\frac{n^{n-\frac{1}{2}}}{2^n} = (n+1).\frac{2n^{n-\frac{1}{2}}}{2^{n+1}} < (n+1).\frac{(n+1)^{n-\frac{1}{2}}}{2^{n+1}} = \frac{(n+1)^{n+\frac{1}{2}}}{2^{n+1}},$$

which proves (2).

Reference:

[1] Mitrinović, D., M. Popadić, Inequalities in Number Theory. Niś, Univ. of Niś, 1978.