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CONTINUED FRACTION REPRESENTATIONS FOR THETA FUNCTIONS

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Abstract. In this paper we establish continued fraction representations for theta functions, which are similar to q-continued fractions for certain q-products recorded by Ramanujan in the unorganized portions of his notebooks.

1. Introduction

Some of Ramanujan's deepest contributions to mathematics lie in the area of q-series and their continued fraction representations. Most of these profound discoveries are found in his "lost" notebook [7]. However, Ramanujan first efforts in this area are found in Chapter 16 of his second notebook [1], [6]. Ramanujan recorded many beautiful continued fraction in the unorganized portion of his notebooks. G. E. Andrews et al. [3] have proved all these continued fraction identities. Some q-continued fractions of Ramanujan give representations for certain q-products. For example

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1+}\,\frac{-q}{1+q+}\,\frac{-q^3}{1+q^2+}\,\frac{-q^5}{1+q^3+}\,\frac{-q^7}{1+q^4+}\cdots, \quad |q|<1, \tag{1.1}$$

and

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty} = \frac{1}{1+}\,\frac{-q}{1+q^2+}\,\frac{-q^3}{1+q^4+}\,\frac{-q^5}{1+q^6+}\,\frac{-q^7}{1+q^8+}\cdots, \quad |q|<1, \tag{1.2}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Proofs of (1.1) and (1.2) may be found in [3].

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In Chapter 16 of his second notebook [1], [6], Ramanujan develops the theory of theta functions and his theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{1.3}$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.4}$$

The product representations of these theta functions can be derived by using the Jacobi triple product identity:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

Recently C. Adiga et al. [2] have obtained some integral representation of theta functions.

In this paper we will establish continued fraction representations of $\varphi(q)$ and $\psi(q)$ which are similar to (1.1) and (1.2).

2. Continued fraction representations for $\varphi(q)$ and $\psi(q)$

Theorem 2.1. *Let $\varphi(q)$ be as defined in (1.3). If $|q| < 1$, then*

$$\varphi(q) = -1 + \frac{2}{1+}\;\frac{-q}{1+q+}\;\frac{-q^3}{1+q^3+}\;\frac{-q^5}{1+q^5+}\;\cdots. \tag{2.1}$$

First Proof. It is well-known that, for $n \geq 1$,

$$\frac{a_1}{b_1+}\;\frac{a_2}{b_2+}\;\frac{a_3}{b_3+}\;\cdots\;\frac{a_n}{+b_n} = \frac{P_n}{Q_n},$$

where

$$P_n = b_n P_{n-1} + a_n P_{n-2} \tag{2.2}$$

and

$$Q_n = b_n Q_{n-1} + a_n P_{n-2}, \tag{2.3}$$

where $P_{-1} = 1 = Q_0$ and $P_0 = 0 = Q_{-1}$ [8, p. 15]. We first note that (2.1) is equivalent to

$$\sum_{n=0}^{\infty} q^{n^2} = \frac{1}{1+} \frac{-q}{(1+q)+} \frac{-q^3}{(1+q^3)+} \frac{-q^5}{(1+q^5)+} \cdots . \tag{2.4}$$

We shall establish (2.4) by proving

$$\frac{1}{1+} \frac{-q}{(1+q)+} \frac{-q^3}{(1+q^3)+} \cdots \frac{-q^{2n-3}}{+(1+q^{2n-3})} = \frac{P_n}{Q_n}, \tag{2.5}$$

where

$$P_1 = 1, \quad P_n = \sum_{k=0}^{n-1} q^{k^2} \quad \text{for } n \geq 2$$

and

$$Q_n = 1.$$

It is obvious that (2.5) holds for $n = 1$ and 2. Proceeding by induction and using (2.2), we find that

$$\begin{aligned} P_n &= (1+q^{2n-3})P_{n-1} - q^{2n-3}P_{n-2} \\ &= (1+q^{2n-3}) \sum_{k=0}^{n-2} q^{k^2} - q^{2n-3} \sum_{k=0}^{n-3} q^{k^2} \\ &= \sum_{k=0}^{n-2} q^{k^2} + q^{(2n-3)+(n-2)^2} \\ &= \sum_{k=0}^{n-1} q^{k^2}. \end{aligned}$$

Similarly using (2.3), we obtain

$$Q_n = (1+q^{2n-3})Q_{n-1} - q^{2n-3}Q_{n-2} = 1.$$

This completes the proof.

Second Proof. From Entry 11 of Chapter 16 in Ramanujan's second notebook [1, p. 14],

$$\frac{\frac{(-a;q)_\infty (b;q)_\infty}{(a;q)_\infty (-b;q)_\infty} - 1}{\frac{(-a;q)_\infty (b;q)_\infty}{(a;q)_\infty (-b;q)_\infty} + 1} = \frac{a-b}{1-q+} \frac{(a-bq)(aq-b)}{1-q^3+} \frac{q(a-bq^2)(aq^2-b)}{1-q^5+} \cdots ,$$

where $|q| < 1, |a| < 1$. After some simplication, the above identity can be written as

$$\frac{(-a;q)_\infty(b;q)_\infty}{(a;q)_\infty(-b;q)_\infty}+1=\frac{2}{1+}\,\frac{b-a}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3+}\,\frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\cdots. \quad (2.6)$$

Changing q to q^2 in (2.6) and then setting $a = q$ and $b = q^2$ we obtain

$$\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}+1=\frac{2}{1+}\,\frac{-q(1-q)}{1-q^2+}\,\frac{-q^3(1-q^3)(1-q)}{1-q^6+}\,\frac{-q^5(1-q^5)(1-q^3)}{(1-q^{10})+}\cdots.$$

Using (1.3), the above identity may be written as

$$\varphi(q)+1=\frac{2}{1+}\,\frac{-q}{1+q+}\,\frac{-q^3}{1+q^3+}\,\frac{-q^5}{1+q^5+}\cdots,$$

which is what we wanted to prove.

Theorem 2.2. *Let $\psi(q)$ be as defined in (1.4). If $|q| < 1$, then*

$$\psi(q)=\frac{1}{1+}\,\frac{-q}{1+q+}\,\frac{-q^2}{1+q^2+}\,\frac{-q^3}{1+q^3+}\cdots. \quad (2.7)$$

First Proof. First, we prove that

$$\frac{1}{1+}\,\frac{-q}{1+q+}\,\frac{-q^2}{1+q^2+}\cdots\frac{-q^{n-1}}{+1+q^{n-1}}=\frac{P_n}{Q_n}, \quad (2.8)$$

where

$$P_1=1, \quad P_n=\sum_{k=0}^{n-1}q^{\frac{k(k+1)}{2}} \quad \text{for } n\geq 2$$

and

$$Q_n=1.$$

It is clear that (2.8) holds for $n = 1$ and 2. Assume that (2.8) holds for all integers $\leq n-1$. Using (2.2), we find that

$$\begin{aligned}
P_n &= (1+q^{n-1})P_{n-1}-q^{n-1}P_{n-2}\\
&= (1+q^{n-1})\sum_{k=0}^{n-2}q^{\frac{k(k+1)}{2}}-q^{n-1}\sum_{k=0}^{n-3}q^{\frac{k(k+1)}{2}}\\
&= \sum_{k=0}^{n-2}q^{\frac{k(k+1)}{2}}+q^{(n-1)+\frac{(n-1)(n-2)}{2}}\\
&= \sum_{k=0}^{n-1}q^{\frac{k(k+1)}{2}}.
\end{aligned}$$

Similarly using (2.3), we obtain

$$Q_n = (1+q^{n-1})Q_{n-1} - q^{n-1}Q_{n-2} = 1.$$

Hence, by induction (2.8) holds for all n. Letting n tend to ∞ in (2.8), we complete the proof of (2.7).

Second Proof. In [4], S. Bhargava and C. Adiga have proved that

$$\frac{G(aq,\lambda q,b;q)}{G(a,\lambda,b;q)} = \frac{1}{1+}\,\frac{aq+\lambda q}{1-aq+bq+}\cdots\frac{aq+\lambda q^n}{1-aq+bq^n+}\cdots, \tag{2.9}$$

where

$$G(a,\lambda,b;q) = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(-\frac{\lambda}{a};q)_n a^n}{(q;q)_n(-bq;q)_n},$$

$$(a;q)_0 = 1$$

and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 1.$$

Letting a tend to 0 and then setting $\lambda = -1, b = 1$ in (2.9), we deduce that

$$\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n}} = \frac{1}{1+}\,\frac{-q}{1+q+}\,\frac{-q^2}{1+q^2+}\cdots\frac{-q^n}{1+q^n+}\cdots. \tag{2.10}$$

Now, employing the q-binomial theorem

$$\frac{(-b;q)_\infty}{(a;q)_\infty} = \sum_{n=0}^{\infty}\frac{(-\frac{b}{a};q)_n a^n}{(q;q)_n}$$

in (2.10), we obtain

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1+}\,\frac{-q}{1+q+}\,\frac{-q^2}{1+q^2+}\cdots\frac{-q^n}{1+q^n+}\cdots$$

which is same as (2.7).

Remarks. Theorem 2.1 and Theorem 2.2 can also be proved directly using an identity due to Euler [5, p. 37]. The special case $a = -1$ of Entry 13 of Chapter 16 in Ramanujan's second notebook [1, p. 22], namely

$$\sum_{k=0}^{\infty} q^{\frac{kk+1}{2}} = \frac{1}{1+}\,\frac{-q}{1+}\,\frac{-(q^2-q)}{1+}\,\frac{-q^3}{1+}\,\frac{-(q^4-q^2)}{1+}\cdots$$

gives another continued fraction representation for $\psi(q)$.

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