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GENERALIZED BERNOULLI POLYNOMIALS & JACKSON'S CALCULUS OF SEQUENCES

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Abstract

This paper considers generalized Bernoulli and exponential functions in the context of factorials formed from the elements of divisibility sequences as in the calculus of sequences of Jackson and Ward. The results for the ordinary integers readily follow. Suggestions for further relevant research with commutative diagrams are included.

1. Introduction

We outline here some formal aspects of generalized Bernoulli sequences in terms of divisibility sequences and Jackson's calculus of sequences [7].

Consider generalized Bernoulli polynomials $B_n(x)$:

$$\frac{tE(xt)}{E(t)-1}=\sum_{n=0}^{\infty}B_n(x)t^n/u_n! \tag{1.1}$$

where

$$E(t)=\sum_{n=0}^{\infty}t^n/u_n!$$

in which the $\{u_n\}$ is a normal divisibility sequence with the property that

$$u_{(s,t)}=(u_s,u_t).$$

We also require that $u_0=0, u_1=1, u_n\neq 0, n>1$, and

$$u_n!=u_nu_{n-1}\ldots u_1, u_0!=1.$$

2. Generalized Bernoulli Polynomials

$B_n(0)=B_n$, a generalized Bernoulli number. We shall also use the operator D_x:

$$D_xx^n=u_nx^{n-1}.$$

We look first at $\Delta B_n(x)$, where Δ is the forward difference operator:

$$\Delta B_n(x)=B_n(x+1)-B_n(x).$$

Now

$$\sum_{n=0}^{\infty}B_n(x+1)t^n/u_n!=\frac{tE((x+1)t)}{E(t)-1}$$
$$=\frac{tE(xt)E(t)}{E(t)-1},$$

so

$$\sum_{n=0}^{\infty} \Delta B_n(x) t^n / u_n! = \frac{tE(xt)E(t) - tE(xt)}{E(t)-1}$$
$$= \frac{tE(xt)(E(t)-1)}{E(t)-1}$$
$$= tE(xt)$$
$$= \sum_{n=0}^{\infty} x^n t^{n+1} / u_n!$$
$$= \sum_{n=1}^{\infty} u_n x^{n-1} t^n / u_n!.$$

Thus we get the rather neat result that

$$\Delta B_n(x) = D_x x^n. \tag{2.1}$$

As with ordinary Bernoulli numbers we set $B_0 = 1, B_{2n+1} = 0, n > 1$, and then we get another result we shall use later:

$$B_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{u_n!} \tag{2.2}$$

where

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{u_n!}{u_k! u_{n-k}!}$$

is a Fontené-Ward binomial coefficient. Gould [4] elegantly extended the work of Ward [10] on these; Ward's generalized sequences were rediscoveries of work of Fontené (see Gould). The generalization consists in systematically replacing the ordinary binomial coefficients by a binomial coefficient to the base u. More particularly, when the sequence of Fibonacci numbers $\{F_n\}$ is used, we get the Fibonacci binomial coefficients of Hoggatt [5].

When $\{u_n\} = \{n\}$, the non-negative integers, we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \binom{n}{k}.$$

When $\{u_n\} = \{\underline{q}_n$, the Fermatian numbers [2] defined by

$$\underline{q}_n = 1 + q + \ldots + q^{n-1}, \underline{q}_0 = 0, .$$

where q may be indeterminate, and we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \begin{bmatrix} n \\ k \end{bmatrix},.$$

the well-known q-binomial coefficient [1]. $\underline{q}_n$ is described as the nth Fermatian of index q.

3. Generalized Exponential Series

Ward [8] has established that a sequence is *normal* when

$$E(t)E(-t) = E(t-t) \\ = E(0) \\ = 1.$$

So

$$t(1-E(t)) = (-t)(E(t)-1)$$

and

$$t(1-E(t)) = tE(t)(E(-t)-1),$$

so that

$$\frac{tE(t)}{E(t)-1} = \frac{-t}{E(-t)-1} \\ = \sum_{n=0}^{\infty} B_n(-t)^n / u_n!.$$

But

$$\frac{tE(t)}{E(t)-1} = E(t)\sum_{n=0}^{\infty} B_n t^n / u_n! \\ = \sum_{m=0}^{\infty} t^m / u_m! \sum_{n=0}^{\infty} B_n t^n / u_n! \\ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left\{ {n \atop k} \right\} B_{n-k} t^n / u_n!$$

Thus, on equating coefficients of t, we obtain

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} B_{n-k} \tag{3.1}$$

since

$$B_{2n+1} = 0, \; n \ge 1.$$

We then have that

$$B_0 = 1,$$

$$B_1 = -\frac{1}{u_2},$$

$$B_2 = \frac{1}{u_2} - \frac{1}{u_3},$$

$$B_3 = 0,$$

$$B_4 = \frac{1}{u_2} - \frac{1}{u_5} - \frac{u_4}{u_2}\left(\frac{1}{u_2} - \frac{1}{u_3}\right),$$

and so on. When $\{u_n\} = \{n\}$, these become

$$B_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$B_4 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} = -\frac{1}{30},$$

and so on, as in Carlitz [2].

4. Generalized Euler-Maclaurin Sum

Analogues of other properties of Bernoulli polynomials can be readily obtained. For instance, the importance of the ordinary Bernoulli numbers comes primarily from the *Euler-Maclaurin Sum-formula* for $\sum m^k$. A generalization of this is

$$\sum_{r=0}^{n-1} r^k = \sum_{j=0}^{k} \frac{n^{k+1-j}}{u_{k+1-j}} \left\{ {k \atop j} \right\} B_j \,. \tag{4.1}$$

Proof:

$$u_k! x \sum_{j=0}^{n-1} E(jx) = u_k! x \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} j^i x^i / u_i$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \frac{u_k!}{u_i!} j^i x^{i+1}$$

and the coefficients of x^{k+1} are $\sum_{j=0}^{n-1} j^k$.

Ward [10] has shown that

$$E(x)E(mx) = E((m+1)x),$$

and so

$$\sum_{j=0}^{n-1} E(jx) = \frac{E(nx) - E(0)}{E(x) - 1}$$

$$u_k! x \sum_{j=0}^{n-1} E(jx) = u_k! \frac{x(E(x) - 1)}{E(x) - 1}$$

$$= u_k! \sum_{j=0}^{\infty} B_j \frac{x^j}{u_j!} \sum_{i=0}^{\infty} n^{i+1} \frac{x^{i+1}}{u_{i+1}!}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{n^{i-j+1} u_k!}{u_j!(u_{i-j+1})} B_j x^{i+1},$$

and the coefficients of x^{k+1} are

$$\sum_{j=0}^{i} \frac{n^{i-j+1} u_k!}{u_j!(u_{i-j+1})} B_j ,$$

as required.

The Euler-Maclaurin formula is also of interest within the context of this paper because it is used in the classic proof of the Staudt-Clausen theorem as expounded by Rado [8].

5. Conclusion

The following might be a source of further research relevant to the topics raised in this paper. Let u be the sequence function:

$$u{:}Z \quad \rightarrow \quad Z$$

so that

$$n \quad \overset{u}{\rightarrow} \quad u_n;$$

Let h represent the highest common factor function:

$$h{:}Z \times Z \quad \rightarrow \quad Z$$

so that

$$n,m \quad \overset{h}{\rightarrow} \quad (n,m).$$

Then the divisibility sequences mentioned in this paper and exemplified by the Fermatian and Fibonacci sequences can be represented in terms of a commutative diagram:

$$\begin{array}{ccc} Z \times Z & \overset{u \times u}{\rightarrow} & Z \times Z \\ h \downarrow & & \downarrow h \\ Z & \rightarrow & Z; \end{array}$$

for example,

$$\begin{array}{ccc} n,m & \overset{u \times u}{\rightarrow} & u_n, u_m \\ h \downarrow & & \downarrow h \\ (n,m) & \rightarrow & u_{(n,m)} = (u_n, u_m). \end{array}$$

A commutative diagram of four sets and functions with such a composition of functions represents a concrete category [1]. It would be of interest to seek a natural transformation of functions between this and other categories either to generalize or clarify the structure of the number theoretic results particularly in relation to the recent work in [11].

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