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ON ONE REMARKABLE IDENTITY RELATED TO FUNCTION $\pi(x)$

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Introduction

By $\mathcal{R}_+$ we denote the set of all positive real numbers and $\mathcal{N} = \{1, 2, ...\}$.
Let

$$g = \{g_n\}_{n=1}^{\infty}$$

be sequence such that:

$$g_n \in \mathcal{R}_+, \tag{a_1}$$

$$(\forall n \in \mathcal{N})(g_n < g_{n+1}), \tag{a_2}$$

$$g \text{ is unbounded.} \tag{a_3}$$

For any $x \in \mathcal{R}_+$ we denote by $\pi(x)$ the number of all terms of g, which are not greater than x.

When x satisfies the inequality

$$0 \leq x < g_1$$

we put

$$\pi(x) = 0.$$

Remark 1: The condition (a_3) shows that the number $\pi(x)$ is always finite for a fixed x.

The main result in this paper is the following

Theorem: Let $a, b \in \mathcal{R}_+$ and $b \geq g_1$. Then the identity

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}) \tag{1}$$

holds.

Remark 2: When

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b})$$

we put in (1) $\sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \bullet$ to be zero, i.e., the right hand-side of (1) reduces to $\pi(\frac{a}{b}).\pi(b)$.

Thus, under the conditions of the above Theorem, identity

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \begin{cases} \pi(\frac{a}{b}).\pi(b), & \text{if } \pi(\frac{a}{g_1}) = \pi(\frac{a}{b}) \\ \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}), & \text{if } \pi(\frac{a}{g_1}) > \pi(\frac{a}{b}) \end{cases} \quad (1')$$

holds.

Proof of the Theorem. First, we note that if $a = 0$, then (1), i.e., (1') holds, since

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}) = \pi(0) = 0$$

and therefore, we may use Remark 2.

For that reason, further we assume that $a > 0$.

First, let us prove (1), i.e., (1') for case $b = g_1$.

Now, we have

$$\pi(b) = \pi(g_1) = 1$$

and

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}).$$

Hence

$$\pi(\frac{a}{b}).\pi(b) = \pi(\frac{a}{g_1}).\pi(g_1) = \pi(\frac{a}{g_1})$$

and (1), resp. (1'), is proved, since the left hand-side of (1') coincides with $\pi(\frac{a}{g_1})$. Then it remains only to consider the case

$$g_1 < b \quad (3)$$

and the proof of the Theorem will be finished.

Let (3) holds. We must consider the alternatives

$$b \le \frac{a}{b} \quad (e1)$$

and

$$b > \frac{a}{b}. \quad (e2)$$

Let (e_1) holds. We shall prove (1) in this case. Inequality (3) implies that interval

$$\alpha \equiv [g_1, b] \quad (4)$$

is well defined. Also, (3) and (e_1) yield

$$\frac{a}{b} < \frac{a}{g_1}. \quad (5)$$

Then (5) implies that the interval

$$\beta \equiv (\frac{a}{b}, \frac{a}{g_1}] \tag{6}$$

is well defined, too. Obviously, $\alpha \cap \beta = \emptyset$ and moreoevr, β lie to the right side of α on the real axis.

Let $g_i, g_j \in g$ $(i \neq j)$ be arbitrary. we introduce $\tau_{i,j}$ putting

$$\tau_{i,j} = g_i . g_j . \tag{7}$$

we denote by $\mathcal{P}$ the set of these $\tau_{i,j}$ defined by (7), for which

$$g_i \in g \cap \alpha, \; g_j \in g \cap \beta$$

and inequality

$$\tau_{i,j} \leq a \tag{8}$$

holds. Then we consider the alternatives:

$$\mathcal{P} = \emptyset \tag{u_1}$$

and

$$\mathcal{P} \neq \emptyset. \tag{u_2}$$

Let (u_1) holds. Then $g \cap \beta = \emptyset$.

Indeed, if we assume that there exists $g_j \in g \cap \beta$, then we obtain

$$\tau_{1,j} = g_1 . g_j \leq g_1 . \frac{a}{g_1} = a,$$

i.e., $\tau_{1,j}$ satisfies (8). Therefore, $\tau_{i,j} \in \mathcal{P}$, since $g_1 \in g \cap \alpha$. Hence

$$\mathcal{P} \neq \emptyset.$$

But the last contradicts to (u_1).

Now, $g \cap \beta = \emptyset$ implies

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}). \tag{9}$$

Moreover, the equality

$$\pi(x) = \pi(\frac{a}{b}) \tag{10}$$

holds for each $x \in \beta$.

Let for $i = 1, 2, ..., \pi(b)$: $x_i = \frac{a}{g_i}$. Then

$$g_1 \leq g_i \leq b$$

and therefore for $i = 1, 2, ..., \pi(b)$:

$$x_i \in [\frac{a}{b}, \frac{a}{g_1}]. \tag{11}$$

Now, (10) and (11) yield

$$\pi(\frac{a}{g_i}) = \pi(\frac{a}{b}). \qquad (12)$$

for each $i = 1, 2, ..., \pi(b)$.

But (12) implies

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b), \qquad (13)$$

which proves (1), because of Remark 2.

The case (u_1) is finished.

let (u_2) holds. Then the inequality

$$\pi(\frac{a}{g_1}) > \pi(\frac{a}{b}) \qquad (14)$$

holds.

Indeed, the assumption that (9) holds, implies

$$g \cap \beta = \emptyset.$$

Hence $\mathcal{P} = \emptyset$. But the last equality contradicts to (u_2).

Now, (14) implies that

$$g \cap \beta \neq \emptyset$$

and that

$$g_{\pi(\frac{a}{b})+k} \in g \cap \beta$$

at least for $k = 1$. Therefore, the sum $\sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \bullet$ from the right hand-side of (1) is well defined.

We shall use the following approach to prove (1) in the case (u_2). First, we denote by $\theta(\alpha, \beta)$ the number of all elements of the set $\mathcal{P}$. Second, we shall calculate $\theta(\alpha, \beta)$ using two different ways. Third, we shall compare the results of these two different calculations and as a result we shall establish (1).

First way of calculation

Let

$$E \equiv \{1, 2, ..., \pi(b)\}.$$

If i describes E, then g_i describes $g \cap \alpha$.

Let $E_1 \subset E$ be the set of those $i \in E$ for which there exist at least oen j, such that $g_j \in g \cap \beta$ and $\tau_{i,j} \in \mathcal{P}$. For each $i \in E_1$ we denote by δ_i the number fo those $g_j \in g \cap \beta$, for which $\tau_{i,j} \in \mathcal{P}$. Then, equality

$$\theta(\alpha, \beta) = \sum_{i \in E_1} \delta_i \qquad (15)$$

holds.

On the other hand, from the definition of these g_j it follows that they belong to interval $(\frac{a}{b}, \frac{a}{g_j}]$. Hence, for $i \in E_1$

$$\delta_i = \pi(\frac{a}{g_i}) - \pi(\frac{a}{b}). \tag{16}$$

Remark 3: From the definitions of δ_i and E_1 it follows that $\delta_i > 0$.

Let $i \in E_2$, where

$$E_2 \equiv E - E_1.$$

Then

$$g \cap (\frac{a}{b}, \frac{a}{g_i}] = \emptyset,$$

because in the opposite case we will obtain that $i \in E_1$, that is impossible, since $E_1 \cap E_2 = \emptyset$.

Hence for $i \in E_2$

$$\pi(\frac{a}{g_i}) = \pi(\frac{a}{b}),$$

i.e., for $i \in E_2$

$$\pi(\frac{a}{g_i}) - \pi(\frac{a}{b}) = 0. \tag{17}$$

Now, (15), (16), and (17) imply

$$\theta(\alpha, \beta) = \sum_{i \in E} (\pi(\frac{a}{g_i}) - \pi(\frac{a}{b})),$$

i.e.,

$$\theta(\alpha, \beta) = \sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) - \pi(\frac{a}{b}).\pi(b). \tag{18}$$

Second way of calculation

Let

$$W \equiv \{\pi(\frac{a}{b}) + k \mid k = 1, 2, ..., \pi(\frac{a}{g_1}) - \pi(\frac{a}{b})\}.$$

Of course, we have $W \neq \emptyset$, since (u_2), i.e., (14), is true.

When j describes W, g_j describes $g \cap \beta$. For every such j it is fulfilled

$$g_1 \leq \frac{a}{g_j} < b. \tag{19}$$

Therefore, there exist exactly $\pi(\frac{a}{g_j})$ in number $g_i \in g \cap \beta$, for which $\tau_{i,j} \in \mathcal{P}$. Hence

$$\theta(\alpha, \beta) = \sum_{j \in W} \pi(\frac{a}{g_j}).$$

Thus, using the definition of W, we finally get

$$\theta(\alpha, \beta) = \sum_{j=1}^{\pi(\frac{a}{g_1}) - \pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}). \tag{20}$$

If we compare (18) and (20), we prove (1) in case (u_2).

Up to now, we have established that (1) (and (1')) holds, when

$$g_1 \le b \le \frac{a}{b} \tag{21}$$

and case (u_2) is finished too.

Now, let (e_2) hold. To prove (1') (and (1)) in this case we consider the alternatives

$$\frac{a}{b} < g_1 \tag{e_{21}}$$

and

$$\frac{a}{b} \ge g_1. \tag{e_{22}}$$

Let (e_{21}) hold. Then $\pi(\frac{a}{b}) = 0$ and (1) looks like

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \sum_{j=1}^{\pi(\frac{a}{g_1})} \pi(\frac{a}{g_j}). \tag{22}$$

We must note, that (21) imlies $b > \frac{a}{g_1}$. Then (22) will be proved, if we prove that for all $k \in \mathcal{N}$

$$\pi(\frac{a}{g_{\pi(\frac{a}{g_1})+k}}) = 0. \tag{23}$$

But $g_{\pi(\frac{a}{g_1})+k} \notin W$. Then we have that $g_{\pi(\frac{a}{g_1})+k} > \frac{a}{g_1}$. Hence, for all $k \in \mathcal{N}$

$$\frac{a}{g_{\pi(\frac{a}{g_1})+k}} < g_1.$$

The last inequalities prove (23), since $\pi(g_1) = 1$ and for $0 \le x < g_1$ it is fulfilled $\pi(x) = 0$. Therefore, (22) is proved, too, and the case (e_{21}) is finished.

Let (e_{22}) holds. Then it is fulfilled

$$g_1 \le \frac{a}{b} < b. \tag{24}$$

We introduce the number b_1 putting

$$b_1 = \frac{a}{b}. \tag{25}$$

Then, we find

$$b = \frac{a}{b_1}. \tag{26}$$

From (24), (25), and (26) it follows immediatelly

$$g_1 \le b_1 < \frac{a}{b_1}. \tag{27}$$

Obviously, (27) looks like (21) (only b from (21) is changed with b_1 in (27)). But we proved that (21) imply (1). Therefore, (27) imply (1), but with b_1 instead of b. Hence, the identity

$$\sum_{j=1}^{\pi(b_1)} \pi(\frac{a}{g_j}) = \pi(\frac{a}{b_1}).\pi(b_1) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b_1})} \pi(\frac{a}{g_{\pi(\frac{a}{b_1})+j}}) \qquad (28)$$

holds and Remark 2 stays valid again after changing b with b_1.

Using (25) we rewrite (28) in the form

$$\sum_{i=1}^{\pi(\frac{a}{b})} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(b)} \pi(\frac{a}{g_{\pi(b)+j}}). \qquad (29)$$

First, let $\pi(b) = \pi(\frac{a}{b})$. In this case (29) coincides with (1) and (1) is proved, since (29) is true.

Second, let $\pi(\frac{a}{b}) < \pi(b)$. Then we add to the two hand-sides of (29) the sum

$$\sum_{j=1}^{\pi(b)-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}})$$

and obtain again (1). This finishes the proof of (1) in this case, too, sence, (29) is true.

Since, we have no other possibilities (the inequality $\pi(b) < \pi(\frac{a}{b})$ is impossible, because (e_2)), we finish with the case (e_{22}). Hence, the case (e_2) is finished too.

The Theorem is proved.

Further, we use some well known functions (see, e.g., [1]):

$$\text{ch}x \equiv \frac{e^x + e^{-x}}{2}, \quad \text{sh}x \equiv \frac{e^x - e^{-x}}{2}, \quad \text{th}x \equiv \frac{\text{sh}x}{\text{ch}x}, \quad \text{cth}x \equiv \frac{\text{ch}x}{\text{sh}x}.$$

Corollary 1: Let $a = \text{ch}x, b = \text{sh}x$, where $x \in \mathcal{R}_+$ and $\text{sh}x \geq g_1$. Then, the identity

$$\sum_{i=1}^{\pi(\text{sh}x)} \pi(\frac{\text{ch}x}{g_i}) = \begin{cases} \pi(\text{sh}x).\pi(\text{cth}x), & \text{if } \pi(\frac{\text{ch}x}{g_1}) = \pi(\text{cth}x) \\ \pi(\text{sh}x).\pi(\text{cth}x) + \sum_{j=1}^{\pi(\frac{\text{ch}x}{g_1})-\pi(\text{cth}x)} \pi(\frac{\text{ch}x}{g_{\pi(\text{cth}x)+j}}), & \text{if } \pi(\frac{\text{ch}x}{g_1}) > \pi(\text{cth}x) \end{cases} \qquad (30)$$

holds.

The same way, putting: $a = \text{sh}x$, $b = \text{ch}x$, where $x \in \mathcal{R}_+$ and $\text{ch}x \geq g_1$, as a corollary of the Theorem, we obtain another identity, that we do not write here since one may get it putting in (30) $\text{ch}x, \text{sh}x, \text{th}x$ instead of $\text{sh}x, \text{ch}x, \text{cth}x$, respectively.

Now, let g be the sequence of all primes, i.e.,

$$g = 2, 3, 5, 7, 11, 13, ...$$

Then the function $\pi(x)$ coincides with the famous function π of the prime number distribution. Thus, from our Theorem we obtain

Corollary 2: Let $a, b \in \mathcal{R}_+$, $b \geq 2$ and $\{p_n\}_{n=1}^{\infty}$ is the sequence of all primes. Then the identity

$$\pi(\frac{a}{p_1}) + \pi(\frac{a}{p_2}) + ... + \pi(\frac{a}{p_{\pi(b)}}) = \pi(\frac{a}{b}).\pi(b) + \pi(\frac{a}{p_{\pi(\frac{a}{b})+1}}) + \pi(\frac{a}{p_{\pi(\frac{a}{b})+2}}) + ...\pi(\frac{a}{p_{\pi(\frac{a}{2})}}) \quad (31)$$

holds.

In (31) $\pi(x)$ denotes (as usually) the number of primes, which are not greater than x. Also, the right hand-side of (31) reduces to $\pi(\frac{a}{b}.\pi(b)$ if and only if $\pi(\frac{a}{b}) = \pi(\frac{a}{2})$.

Finally, we must observe that the main result of the present paper was discovered trough the year 2001 in Sinemoretz (a Bulgarian vilagge on Black See).

REFERENCE:

[1] Handbook of Mathematical Functions, M. Abramowitz and I. Stegun (Eds.), National Bureau of Standards, Applied Mathematics Series, Vol. 55, 1964. 1960.