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ON THE VALUES OF p-ADIC q-L-FUNCTIONS, II

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ABSTRACT. In the recent paper, we defined the h-extension of q-Bernoulli number by using multiple p-adic q-integral and constructed the h-extension of complex analytic q-L-series which interpolates the h-extension of q-Bernoulli numbers, cf. [2], [4], [5]. The purpose of this paper is to construct a h-extension of p-adic q-L-function which interpolates the h-extension of q-Bernoulli numbers at non-positive integers.

1. INTRODUCTION

In 1982, Koblitz constructed p-adic q-L-function which interpolates Carlitz's q-Bernoulli number at non-positive integers and suggested two questions. Question 1 was solved by Satoh (see [8]) and Question 2 was solved by T. Kim. Satoh constructed a complex analytic q-L-series which is a q-analogue of Drichlet's L-function and interpolates q-Bernoulli number, which is an answer to Koblitz's question (see [7]). In

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[2], we constructed p-adic q-integral and proved that Carlitz's q-Bernoulli numbers can be represented as an p-adic q-integral . Let h be a fixed positive integer. In recently, we defined the h-extension of q-Bernoulli number by using multiple p-adic q-integral and constructed h-extension of complex analytic q-ζ-series which interpolates the h-extension of q-Bernoulli number as follows (see [2]): For $s \in \mathbb{C}$, define

$$\zeta_q^{(h)}(s) = \frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]^{s-1}} + \sum_{n=1}^{\infty}\frac{q^{nh}}{[n]^s}.$$

It is easy to see that $\zeta_q^{(h)}$ is meromorphic function on $\mathbb{C}$ with only one simple pole at $s=1$. In [2], the h-extension of complex analytic q-L-series $L_q^{(h)}(s,\chi)$ was also defined by author. Note that we can recover the results of Satoh at $h=1$. However, we did not construct the h-extension of p-adic q-L-function which interpolates the h-extension of q-Bernoulli number at non-positive integers yet now.

$$\begin{array}{ccccc}
\zeta(s) & \longrightarrow & \overset{\text{complex}}{L(s,\chi)} & \longrightarrow & \overset{p-\text{adic}}{L_p(s,\chi)} \\
\downarrow & & \downarrow & & \downarrow \\
\underset{\text{(see [8])}}{\zeta_q(s)} & \longrightarrow & \underset{\text{(see [8])}}{L_q(s,\chi)} & \longrightarrow & \underset{\text{(see [7])}}{L_{p,q}(s,\chi)} \\
\downarrow & & \downarrow & & \downarrow \\
\underset{\text{(see [2])}}{\zeta_q^{(h)}(s)} & \longrightarrow & \underset{\text{(see [2])}}{L_q^{(h)}(s,\chi)} & \longrightarrow & ?
\end{array}$$

The purpose of this paper is to construct the h-extension of p-adic q-L-function $L_{p,q}^{(h)}(s,\chi)$ which $L_{p,q}^{(h)}(s,\chi)$ interpolates $L_q^{(h)}(s,\chi)$. Note that we can recover the theorem of Kobiltz at $h=1$ (see [7]).

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2. Some p-adic q-integrals

Let p be a fixed prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, \ \ X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$ (see [3]).

The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assumes $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

Throughout this paper, we use the following notation :

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]} \sum_{0 \le j < p^N} q^i f(j) = \sum_{0 \le j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p)$$

representing q-analogue of Riemann sums for f.

The integral of f on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of these sums, when it exists. The p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{0 \le j < p^N} f(j) q^j.$$

Note that if $f_n \to f$ in $UD(Z_p)$; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

In the recent paper (see [2]), we have defined q-Bernoulli number of higher order as follows:

$$\beta_m^{(h,k)} = \beta_m^{(h,k)}(q) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x_1 + \cdots + x_k]^m q^{\sum_{i=1}^k x_i(h-i)} d\mu_q(x_1) \cdots d\mu_q(x_k),$$

where k, h are positive integers.

In this paper we only consider $\beta_m^{(h,1)}$, so $\beta_m^{(h)}$ rather than $\beta_m^{(h,1)}$. By the definition of $\beta_m^{(h)}$, that is, $\beta_m^{(h)} = \beta_m^{(h)}(q) = \int_{\mathbb{Z}_p} [x]^m q^{x(h-1)} d\mu_q(x)$, we see:

$$\beta_0^{(h)} = \frac{h}{[h]}, \quad q^h(q\beta^{(h)}+1)^m - \beta_m^{(h)} = \begin{cases} 1 & if \quad m=1 \\ 0 & if \quad m>1 \end{cases},$$

with the usual convention about replacing $(\beta^{(h)})^i$ by $\beta_i^{(h)}$.

In [2], the "h-extension of q-Bernoulli polynomials" $\beta_m^{(h)}(x,q)$ were defined by

$$\beta_m^{(h)}(x,q) = \int_{\mathbb{Z}_p} [x+t]^m q^{t(h-1)} d\mu_q(t),$$

which can be written as $(q^x\beta^{(h)} + [x])^m = \beta_m^{(h)}(x,q)$, for $m \geq 1$.

The h-extension of q-Bernoulli polynomials $\beta_m^{(h)}(x,q)$ satisfy the following generalized distribution relation:

$$[l]^{m-1}\sum_{i=0}^{l-1} q^{ih}\beta_m^{(h)}\left(\frac{x+i}{l}, q^l\right) = \beta_m^{(h)}(x,q), \tag{1}$$

for any positive integer m, l.

We use (1) to define p-adic distributions, then "regularize" to get bounded measures, and finally take the Mellin transform to define the h-extension of p-adic q-L-function which interpolate the h-extension of q-Bernoulli numbers.

3. h-EXTENSION OF p-ADIC q-L-FUNCTIONS

For $k \geq 1$, let $\mu_k^{(h)} = \mu_{k:q}^{(h)}$ be defined by

$$\mu_k^{(h)}(a + dp^N\mathbb{Z}_p) = [dp^N]^{k-1} q^{ha} \beta_k^{(h)}\left(\frac{a}{dp^N}, q^{dp^N}\right).$$

Then we easily see that $\mu_k^{(h)}$ extends to a $\mathbb{Q}_p(q)$-valued distribution on the compact open set $U \subset X$ by using (1) (see [3]).

Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{Z}_{\geq 0}$. For $h \geq 0$, define

$$\beta_{m,\chi}^{(h)} = \beta_{m,\chi}^{(h)}(q) = \int_X q^{(h-1)x}\chi(x)[x]^m d\mu_q(x).$$

Note that

$$\beta^{(h)}_{m,\chi} = [d]^{m-1}\sum_{i=1}^{d-1}\chi(i)q^{hi}\beta^{(h)}_m(\frac{i}{d}, q^d).$$

Let $\alpha \in X^*, \alpha \neq 1, k \geq 1$. By the definition of $\mu^{(h)}_k$, we easily see:

$$\int_X \chi(x)d\mu^{(h)}_k(x) = \beta^{(h)}_{k,\chi},$$

$$\int_{pX} \chi(x)d\mu^{(h)}_k(x) = [p]^{k-1}\chi(p)\beta^{(h)}_{k,\chi}(q^p),$$

$$\int_X \chi(x)d\mu^{(h)}_{k,q^{\frac{1}{\alpha}}}(\alpha x) = \chi(\frac{1}{\alpha})\beta^{(h)}_{k,\chi}(q^{\frac{1}{\alpha}}),$$

$$\int_{pX} \chi(x)d\mu^{(h)}_{k,q^{\frac{1}{\alpha}}}(\alpha x) = [p : q^{\frac{1}{\alpha}}]^{k-1}\chi(\frac{p}{\alpha})\beta^{(h)}_{k,\chi}(q^{\frac{p}{\alpha}}). \tag{2}$$

For compact open set $U \subset X$, define

$$\mu^{(h)}_{k,\alpha}(U) = \mu^{(h)}_{k,\alpha:q}(U) = \mu^{(h)}_{k:q}(U) - \alpha^{-1}[\alpha^{-1} : q]^{k-1}\mu^{(h)}_{k:q^{\frac{1}{\alpha}}}(U\alpha).$$

By the definition of $\mu^{(h)}_{k,\alpha}$ and (2), note that

$$\begin{aligned}\int_{X^*} \chi(x)d\mu^{(h)}_{k,\alpha}(x) &= \beta^{(h)}_{k,\chi} - [p]^{k-1}\chi(p)\beta^{(h)}_{k,\chi}(q^p) - \frac{1}{\alpha}[\frac{1}{\alpha}]^{k-1}\chi(\frac{1}{\alpha})\beta^{(h)}_{k,\chi}(q^{\frac{1}{\alpha}})\\ &\quad + \frac{1}{\alpha}[\frac{p}{\alpha}]^{k-1}\chi(\frac{p}{\alpha})\beta^{(h)}_{k,\chi}(q^{\frac{p}{\alpha}})\\ &= (1-\chi^p)(1-\frac{1}{\alpha}\chi^{\frac{1}{\alpha}})\beta^{(h)}_{k,\chi},\end{aligned} \tag{3}$$

where the operator $\chi^y = \chi^{y,k:q}$ on $f(q)$ is defined by

$$\chi^y f(q) = [y]^{k-1}\chi(y)f(q^y), \quad \chi^x\chi^y = \chi^{x,k:q^y}\circ\chi^{y,k:q}.$$

If for $x \in X$ we let $\{x\}_N$ denote the least nonnegative residue (mod dp^N) and let $[x]_N$ denote $x - \{x\}_N$, so that $[x]_N \in dp^N\mathbb{Z}_p$.

Now, we can define in [3] as follows:

$$\mu^{(h)}_{Mazur,1,\alpha} = \left(\frac{\frac{1}{\alpha}-1}{h+1} + \frac{h}{\alpha}\frac{[\alpha a]_N}{dp^N}\right).$$

By the same method of Koblitz (see [7]), we easily see:

$$\lim_{N\to\infty}\mu_{k,\alpha}^{(h)}(a+dp^N\mathbb{Z}_p)$$
$$=\lim_{N\to\infty}\{[a]^{k-1}((h+k)q^{(h+1)a}-hq^a)(\frac{(\frac{1}{\alpha}-1)}{h+1}+\frac{[a\alpha]_N}{dp^N}\frac{h}{\alpha})\}.$$

Thus we have

$$d\mu_{k,\alpha}^{(h)}(x)=[x]^{k-1}((h+k)q^{(h+1)x}-hq^{xh})d\mu_{Mazur,1,\alpha}^{(h)}(x). \tag{4}$$

Note that $\mu_{k,\alpha}^{(h)}$ are bounded $\mathbb{Q}_p$-valued measure on X for all $k\geq 1$ and $\alpha\in X^*,\alpha\neq 1$.

Now, we define $<x>=<x:q>=[x:q]/w(x)$, where $w(x)$ is the Teichmüller character. For $|q-1|_p<p^{-\frac{1}{p-1}}$, note that $<x>^{p^N}\equiv 1 \pmod{p^N}$. By (3),(4), we obtain the following:

$$\int_{X^*}\chi_k(x)d\mu_{k,\alpha}^{(h)}(x)=\int_{X^*}((h+k)q^{(h+1)x}-hq^{xh})<x>^{k-1}\chi_1(x)d\mu_{Mazur,1,\alpha}^{(h)}(x), \tag{5}$$

where $\chi_k=\chi w^{-k}$.

By using (5), we can construct h-extension of p-adic q-L-function as follows:

(h-EXTENSION OF p-ADIC q-L-FUNCTIONS)

For fixed $\alpha\in X^*,\alpha\neq 1$, define

$$L_{p,q}^{(h)}(s,\chi)=\frac{1}{s-1}\int_{X^*}((h+1-s)q^{(h+1)x}-hq^{xh})<x>^{-s}\chi_1(x)d\mu_{Mazur,1,\alpha}^{(h)}(x),$$

for $s\in\mathbb{Z}_p$.

Note that

$$L_{p,q}^{(h)}(1-k,\chi)=-\frac{1}{k}(1-\chi_k^p)(1-\frac{1}{\alpha}\chi_k^{\frac{1}{\alpha}})\beta_{k,\chi_k}^{(h)}.$$

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