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HOW FERMAT'S GREAT THEOREM HELPS SOLVING THE DIOPHANTINE EQUATION $12x^3 - \varphi(y).y^2 = 3.(\varphi(y))^3$

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In the title of the paper φ denotes the famous Euler's totient function.

Except

$$12x^3 - \varphi(y).y^2 = 3.(\varphi(y))^3, \quad (1)$$

in this note the Diophantine equations

$$12x^3 - ky^2 = 3k^3, \quad (2)$$

$$x^3 - 6ky^2 = 2k^3, \quad (3)$$

have also been studied with k being an integer parameter.

A special trick is used to solve each of the above equations. The trick is based on Fermat's Great Theorem, especially for the case $n = 3$, which was proved by L. Euler (see [1]). Of course, the same tricks are possible for the cases $n = 4, 5, 6, ...$, too, but with respect to suitably chosen Diophantine equations of n-th power.

The author used this approach for the first time in [2] to solve the Diophantine equation

$$12x^3 - y^2 = 3 \quad (3)$$

and some others Diophantine equations connected with it.

The first result in the present paper is

Theorem 1. All integer solutions (x, y) of (1) are given by

$$x = 2^\alpha.3^{\beta-1}, \ y = 2^\alpha.3^\beta,$$

where α and β run the set of all positive integers.

Proof: Let the couple (x, y) be an arbitrary integer solution of (1). Then we note that $x \neq 0$. Also, we have that y is a positive integer, because Euler's function is defined for positive integers only.

The trick mentioned above is the following.

We introduce three new numbers u, v, w, using the substitutions

$$u = y - 3.\varphi(y), \ v = 6.x, \ w = y + 3.\varphi(y). \quad (6)$$

Obviously, we have $v \neq 0$, $w \neq 0$, because of $x \neq 0$, $y > 0$. Also, u, v, w are integers. Moreover, one may verify that these numbers satisfy the equality

$$u^3 + v^3 = w^3. \tag{7}$$

The last relation follows from the fact that the couple (x, y) is supposed to be a solution of (1). But if we have $u \neq 0$, then (7) contradicts to Fermat's Great Theorem (for the case $n = 3$). Therefore, $u = 0$. Hence $v = w$ and as a result we obtain

$$y = 3.\varphi(y), \tag{8}$$

$$x = \varphi(y), \tag{9}$$

Let us consider (8). We conclude that $y > 2$, since $y = 1$ and $y = 2$ are not solutions of (8). Hence $\varphi(y)$ and y are even numbers. Therefore,

$$y = 2^{m_1}.\prod_{i=2}^{k} p_i^{m_i}, \tag{10}$$

where $k \geq 2$ is an integer, p_i $(i = 2, 3, ..., k)$ are different primes greater than 2, and m_i $(i = 1, 2, ..., k)$ are positive integers.

We must note the that case $y = 2^{m_1}$ is impossible, because of (8).

Using (10) and Euler's formula

$$\varphi(y) = 2^{m_1 - 1}.\prod_{i=2}^{k} p_i^{m_i - 1}.(p_i - 1),$$

we obtain

$$2.\prod_{i=2}^{k} p_i = 3.\prod_{i=2}^{k}(p_i - 1).$$

Let us denote by H_l the left and by H_r the right side of (11). Obviously, if $k > 2$, we have

$$H_r \equiv 0(mod 4),$$

but the same congruence is not fulfilled for H_l. So, (11) is impossible for $k > 2$. Therefore, $k = 2$ and $y = 2^{m_1}.p_2^{m_2}$. From (8) it follows that $p_2 = 3$. Finally, we get

$$y = 2^{m_1}.3^{m_2} \tag{12}$$

and (9) immediately yields from (12)

$$x = 2^{m_1}.3^{m_2 - 1}. \tag{13}$$

From (12) and (13) we obtain (5) by substituting $m_1 = \alpha$ and $m_2 = \beta$.

Now, let the couple (x, y) be given by (5). In this case one may easyly verify that (1) holds and Theorem 1 is proved.

The second result is

Theorem 2: All integer solutions (x, y) of (2) are given by $x = k$ and $y = \pm 3.k$.

To prove this Theorem we must substitute $u = y - 3.k$, $v = 6.x$, $w = y + 3.k$ and observe that if (x, y) is a solution of (2), then we have again $u^3 + v^3 = w^3$.

The third result is

Theorem 3: All integer solutions (x, y) of (3), when $k \neq 0$, are given by $x = 2.k$ and $y = \pm k$.

To prove this Theorem we substitute in (3): $x = 2.k.a$ and we obtain $y = k.b$. Hence,

$$4.a^2 - 3.b^2 = 1. \tag{14}$$

But as a corollary of Theorem 2, in the case $k = 1$, it follows that all integer solutions of (14) are $a = 1$ and $b = \pm 1$. Therefore, $x = 2.k$ and $y = \pm k$ are all integer solutions of (3).

We must note that if the couple (a, b) is a solution of (14), then numbers $u = b - 1$, $v = 2.a$, $w = b + 1$ satisfy the equation $u^3 + v^3 = w^3$. From here, as in the previous case, we conclude again that all integer solutions of (14) are $a = 1$ and $b = \pm 1$.

Also, as a corollary of Theorem 2, when $k = 1$, we obtain that all integer solutions of (4) are $x = 1$ and $y = \pm 3$.

References:

[1] Edvards, H. Fermat's Last Theorem. Springer-Verlag, New York, 1977.

[2] Vassilev, M. How to solve the Diophantine equation $A^3 + (A+1)^3 + (A+2)^3 = B^3$. *Bull. of Number Theory and Related Topics*, , Vol. X, 1986, 27-31.