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The Analysis of Twin Primes Within $\mathbb{Z}_6$

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Abstract:
The modular ring $\mathbb{Z}_6$ defines integers via $(6r_i + (i-3))$ where i is the Class and r_i the row when tabulated in an array. Since only Classes $\bar{2}_6$ and $\bar{4}_6$ contain odd primes, this modular ring is ideally suited for the analysis of twin primes. In considering a series of integers, a simple method is used to calculate rows (F rows) that do not contain twin primes. This allows the distribution of other primes to be found. Then, in considering the corresponding array of rows, elimination of the F rows yields the rows which contain twin primes. The calculations are facilitated by the use of the right-end digit (*RED*) technique.

1. Introduction

Twin primes are pairs of primes $\{p, p+2\}$ that are known to occur up to very high numbers, such as

$$242206083 \times 2^{3880} \pm 1,$$

a discovery due to K-H Indlekofer (Forbes, 1997). The distribution of twin primes and whether their supply is infinite are unsolved problems (Dunham, 1994) and part of the larger problem of the density of prime numbers (Boyer, 1968). Twin primes can be thus part of a broader prime number research agenda (Holben and Jordan, 1968; Ramachandra, 1998). Properties of sequences of twin primes may be found in Abramowitz and Stegun (1964) and Ribenboim (1989).

Here we analyse the structure of twin-primes distributed within the modular ring $\mathbb{Z}_6$. This allows the twin primes to be represented by an integer, the row, in a tablular representation of $\mathbb{Z}_6$. Rows which do not contain twin primes are easily identified and hence the remaining rows must contain the twins. Calculations can be simplified by using a right end-digit classification system.

In effect, using primes, we build up integers within rows of the modular ring $\mathbb{Z}_6$ in such a way that the structure of each row is revealed. This in turn gives the distribution of primes and allows the extraction of rows which contain twin primes. As well, the underlying structure of the $\mathbb{Z}_6$ integer array is revealed, and this should be of general use in seeking to understand the basis of a wide range of tests of primality and compositeness (Riesel, 1994).

2. Twin Primes in the Modular Ring $\mathbb{Z}_6$

The best modular ring for analysing twin primes is $\mathbb{Z}_6$ (Leyendekkers *et al*, 1995,7). Integers in $\mathbb{Z}_6$ may be represented by$(6r_i + (i-3))$ in which $\bar{i}$ is the class and r is the row. Only Classes $\bar{2}_6$ and $\bar{4}_6$ contain odd primes. In the other odd number class, $\bar{6}_6$, all the integers, n, are divisible by 3.

The advantage of $\mathbb{Z}_6$ is that the twin primes (p_1, p_2) occur in the same row of the tabular array with $p_1 = (6r_2 - 1)$ and $p_2 = (6r_4 + 1)$ and with $r_2 = r_4 = R'$, so that $(p_1 + p_2) = 6(2R')$.Thus the sum of twin primes is an even integer $N \in \bar{3}_6$ in an even row. The integer separating the twin primes (*tps*) will be $\frac{1}{2}N$, since $\frac{1}{2}N - 1 + \frac{1}{2}N + 1 = N$. The row which contains $\frac{1}{2}N$ is $R' \in \bar{3}_6$, so $\frac{1}{2}N = 6R'$.

If either of the integers in $\bar{2}_6$ or $\bar{4}_6$ is composite, then twin primes will not occur. The rows which contain composite integers (designated as F or forbidden rows) are easily obtained as follows.

A composite odd integer, M, is given by(Leyendekkers & Rybak, 1995)

$$M = p^2 + 2p(s-1) \tag{2.1}$$

where p is a prime and $s = 1,2,3,4,...$

Obviously $p|M$, so that for integers $M_2 \in \bar{2}_6$,

$$\begin{aligned} M_2 &= 6r_2 - 1 \\ &= p^2 + 2p(s-1) \end{aligned}$$

so that

$$r_2 = [\tfrac{1}{2}(p^2+1) + p(s-1)]/3 \tag{2.2}$$

and for composite integers in $\bar{4}_6$

$$\begin{aligned} M_4 &= 6r_4 + 1 \\ &= p^2 + 2p(s-1) \end{aligned}$$

so that

$$r_4 = [\tfrac{1}{2}(p^2-1) + p(s-1)]/3 \tag{2.3}$$

and $p \neq 3$ since all $M : 3|M$ are in $\bar{6}_6$. The F rows are therefore given by

$$R = R_0 + pt \tag{2.4}$$

with $t = 0,1,2,3,...$ R_0 functions are listed in Table 1.

Class of M	Class of p	s_0	R_0
$\bar{2}_6$	$\bar{2}_6$	2	$(\frac{1}{2}(p^2+1)+p)/3$
	$\bar{4}_6$	3	$(\frac{1}{2}(p^2+1)+2p)/3$
$\bar{4}_6$	$\bar{2}_6$ & $\bar{4}_6$	1	$(p^2-1)/6$

Table 1: R_0 Functions

Examples of F rows for $4 < R \leq 100$ are given in Table 2. As can be seen, only seven primes are needed to build up the composite integers in the first 100 rows.

The remaining rows will thus contain *tps*; these rows, R', are: 1,2,3,5,7,10,12,17,18,23,25,30,32,33,38, 40,45,47,52,58,70,72,77,87,95,100. The values of the *tps* are simply obtained from $6R' \pm 1$.

The rows can be thought of as slots in the six (infinite) columns of the $\mathbb{Z}_6$ integer array. As the period of a prime brings it around into a slot again (p jumps) this p will have to join with primes already there to form composite integers. Some slots, however, remain empty until new primes occur in them. In effect, Table 3 displays the row and integer anatomy of the modular ring $\mathbb{Z}_6$ for the first 300 odd integers. The character of each F row, in terms of odd integers relatively prime to 3, is clearly shown.

When the recurrence of a row is unity, one of the class $(\bar{2}_6, \bar{4}_6)$ integers must be a prime, in the class occupied by M. The classes of M and p are obtained from Equations (2.2) and (2.3) as illustrated in Table 2.

When the F rows occur more than once a prime will appear provided M is confined to one class. For example, from Table 3, row 27 occurs only once and has M in Class $\bar{2}_6$ with a factor 7. Thus Class $\bar{4}_6$ must contain a prime (given by $6 \times 27 + 1 = 163$). Row 79 occurs three times and both Classes $\bar{2}_6$ and $\bar{4}_6$ are occupied by composite integers which have the factors $11 \in \bar{2}_6$ and $5, 19$ for $M \in \bar{4}_6$. On the other hand, row 99 occurs three times but Class $\bar{2}_6$ is not occupied by any M.

For $R > 1$, the right end digit (*RED*) of a row which contains twin primes can only be $0, 2, 8$ or $3, 5, 7$. This is easily understood from Table 4. When $R^* \in \{1, 4, 6, 9\}$, one of the odd integers in the row for Classes $\bar{2}_6$ or $\bar{4}_6$ must have the *RED* equal to 5, so that row cannot contain a twin prime. When the $RED \neq 5$ for the integer, the *RED*s occur as couples (9,1),(1,3) or (7,9).

$s \downarrow \backslash \underset{\rightarrow}{p}$	5	11	17	23
2	6	24	54	96
5	11	35	71	119
8	16	46	88	.
11	21	57	105	.
14	26	68	.	.
17	31	79	.	.
20	36	90	.	.
23	41	101	.	.
26	46	.	.	.
29	51	.	.	.
32	56	.	.	.
35	61	.	.	.
38	66	.	.	.
41	71	.	.	.
44	76	.	.	.
47	81	.	.	.
50	86	.	.	.
53	91	.	.	.
56	96	.	.	.

Table 2(a): Forbidden Rows for $M \in \bar{2}_6$: Class $\bar{2}_6$ Primes

$s \downarrow \backslash p \rightarrow$	7	13	19
3	13	37	73
6	20	50	92
9	27	63	111
12	34	76	.
15	41	89	.
18	48	102	.
21	55	.	.
24	62	.	.
27	69	.	.
30	76	.	.
33	83	.	.
36	90	.	.
39	97	.	.
42	.	.	.

Table 2(b): Forbidden Rows for $M \in \bar{2}_6$: Class $\bar{4}_6$ Primes

Interestingly the sum *tps* gives an even integer with a *RED* that is compatible with the *RED* of a square. However, rows which contain squares in $\bar{3}_6$ equal $6j^2, j = 1,2,3,...$ Thus, if the sum of twin primes equals a square, $R = 3j^2$, then $(R/3)^{1/2} \in \mathbb{Z}$. Of the 109 rows which contain twin primes for $500 \leq R \leq 1500$, only two fulfil this requirement, namely, 588 and 675 (as in Section 3).

When sorting for twin primes with Equation (2.4) only specific values of the *RED* for t will yield $R^* = 0,2,8,3,5,7$. Thus only a limited number of t^* need to be considered. Tables 5(a) and (b) have t^* values that give $R^* = 0,2,8,3,5,7$.

3. Generalization for R>100

Any range of R can be analysed in the above manner. As an illustration we consider the range of R as 500 – 1500. For this one thousand-row range the complete set of odd integers, prime to 3, is obtained from just 22 primes. The primes and ranges of t (for Equation (2.4)) which are needed to calculate the forbidden rows are shown in Table 6.

The frequency of the F rows and the classes in which the corresponding M values occur (either $\bar{2}_6$ and/or $\bar{4}_6$) indicate all the primes. Primes must occur when only one of the classes shows up in the F row analyses even though that class appears with the F row more than once (Table 3).

If only the *tps* are required then selected values of t (as in Table 5) are used. All rows with *RED*s 0, 2, 8 or 3, 5, 7 that do not appear as F rows will therefore contain *tps*.

Table 7 lists the rows which contain *tps* for the range $500 \leq R \leq 1500$.

$s \downarrow \backslash \underset{\rightarrow}{p}$	5	7	11	13	17	19	23	29
1	4	8	20	28	48	60	88	140
4	9	15	31	41	65	79	111	.
7	14	22	42	54	82	98	.	.
10	19	29	53	67	99	117	.	.
13	24	36	64	80	.	.	.	.
16	29	43	75	93	.	.	.	.
19	34	50	86	106	.	.	.	.
22	39	57	97	.	.	.	.	.
25	44	64	108	.	.	.	.	.
28	49	71	.	.	.	.	.	.
31	54	78	.	.	.	.	.	.
34	59	85	.	.	.	.	.	.
37	64	92	.	.	.	.	.	.
40	69	99	.	.	.	.	.	.
43	74	106	.	.	.	.	.	.
46	79	.	.	.	.	.	.	.
49	84	.	.	.	.	.	.	.
52	89	.	.	.	.	.	.	.
55	94	.	.	.	.	.	.	.
58	99	.	.	.	.	.	.	.

Table 2(c): Forbidden Rows for $M \in \overline{4}_6$

R^*	1	9	4	6	0	2	8	3	5	7
$(6r_2 - 1)^*$	5	3	3	5	9	1	7	7	9	1
$(6r_4 + 1)^*$	7	5	5	7	1	3	9	9	1	3

Table 4: Right End Digits

Row	Recurrence	Class of M	Factor p	Row	Recurrence	Class of M	Factor p
4	1	4	5	34	.	4	5
6	1	2	5	35	1	2	11
8	1	4	7	36	2	2	5
9	1	4	5	.	.	4	7
11	1	2	5	37	1	2	13
13	1	2	7	39	1	4	5
14	1	4	5	41	3	2	5,7
15	1	4	7	.	.	4	13
16	1	2	5	42	1	4	11
19	1	4	5	43	1	4	7
20	2	2	7	44	1	4	5
.	.	4	11	46	2	2	5,11
21	1	2	5	48	2	2	7
22	1	4	7	.	.	4	17
24	2	2	11	49	1	4	5
.	.	4	5	50	2	2	13
26	1	2	5	.	.	4	7
27	1	2	7	51	1	2	5
28	1	4	13	53	1	4	11
29	2	4	5,7	54	3	2	17
31	2	2	5	.	.	4	5,13
.	.	4	11	55	1	2	7
34	2	2	7	56	1	2	5

Table 3(a): Frequency of Forbidden Rows

p^*	t^*
1,9	0,2,8,3,5,7
3	0,4,8,3,5,9
7	0,2,6,1,5,7

Table 5(a): Class of $M = \overline{4}_6$

$p^* \in \overline{2}_6$	t^*	$p^* \in \overline{4}_6$	t^*
3,7	2,4,8,3,7,9	3,7	0,2,6,1,5,7
9	0,2,8,3,5,7	9	0,6,8,1,3,5
1	4,6,8,1,3,9	1	2,4,6,1,7,9

Table 5(b): Class of $M = \overline{2}_6$

Row	Recurrence	Class of M	Factor p	Row	Recurrence	Class of M	Factor p
57	2	2	11	80	1	4	13
.	.	4	7	81	1	2	5
59	1	4	5	82	1	4	17
60	1	4	19	83	1	2	7
61	1	2	5	84	1	4	5
62	1	2	7	85	1	4	7
63	1	2	13	86	2	2	5
64	3	4	5,7,11	.	.	4	11
65	1	4	17	88	2	2	17
66	1	2	5	.	.	4	23
67	1	4	13	89	2	2	13
68	1	2	11			4	5
69	2	2	7	90	2	2	7,11
.	.	4	5	91	1	2	5
71	3	2	5,17	92	2	2	19
.	.	4	7	.	.	4	7
73	1	2	19	93	1	4	13
74	1	4	5	94	1	4	5
75	1	4	11	96	2	2	5,23
76	3	2	5,7,13	97	2	2	7
78	1	4	7	.	.	4	11
79	3	2	11	98	1	4	19
.	.	4	5,19	99	3	4	5,7,17

Table 3(b): Frequency of Forbidden Rows

4. Concluding Comments

As noted previously (Leyendekkers & Shannon, 2000), the sum of $\frac{1}{p}$, where p is a twin prime, taken over all twin primes, converges to Brun's Constant, B; that is,

$$B = 4\sum_{i=1}^{\infty} \frac{N_i}{N_i^2 - 4}$$

$$= 12\sum_{i=1}^{\infty} \frac{R_i}{36R_i^2 - 1}$$

$$\approx 1.9022,$$

where N_i is the sum of the two primes in the ith twin prime set and R_i is the row in $\mathbb{Z}_6$

which contains the ith twin prime set. The difference between the non converging sum

$$4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\bullet\bullet\bullet\right)$$

or π and B is approximately $\frac{7616}{6145}$ or 1.2393816. (For π, $\frac{1}{N} < 0$ when $N \in \bar{3}_4 \subset \mathbb{Z}_4$.)

Ranges of R to be analysed can be kept quite small so that even though more primes are needed for large numbers, the corresponding range of t will not be excessive.

Other constellation primes are more precisely defined in Leyendekkers and Shannon (2000).

Within $\mathbb{Z}_6$, the triple $(p, p+2, p+6)$ always follows the ordered triple $(\bar{2}_6, \bar{4}_6, \bar{2}_6)$; that is, this triple always starts with a Class $\bar{2}_6$ prime. On the other hand, the triple $(p, p+4, p+6)$ constellation follows the order $(\bar{4}_6, \bar{2}_6, \bar{4}_6)$ and cannot start with a Class $\bar{2}_6$ prime. The quadruple $(p, p+2, p+6, p+8)$ starts with a Class $\bar{2}_6$ prime and follows $(\bar{2}_6, \bar{4}_6, \bar{2}_6, \bar{4}_6)$. For related work by A Schinzel see Halberstam and Richert (1974).

$M \in \bar{4}_6$			$M \in \bar{2}_6$		
p	R_0	Range of t	p	R_0	Range of t
5	4	100,299	5	6	99,228
7	8	71,213	11	24	44,134
11	20	44,134	17	54	27,85
13	28	37,113	23	96	18,61
17	48	27,85	29	150	13,46
19	60	24,75	41	294	6,29
23	88	18,61	47	384	3,23
29	140	13,46	53	486	1,19
31	160	11,43	59	600	0,15
37	228	8,34	71	864	0,8
41	280	6,29	83	1176	0,3
43	308	5,27	89	1350	0,1
47	368	3,24	7	13	70,212
53	468	1,19	13	37	36,112
59	580	0,15	19	73	23,75
61	620	0,14	31	181	11,43
67	748	0,11	37	253	7,33
71	840	0,9	43	337	4,27
73	888	0,8	61	661	0,13
79	1040	0,5	67	793	0,10
83	1148	0,4	73	937	0,7
89	1320	0,2	79	1093	0,5

Table 6: Data for the Calculation of Forbidden Rows, $500 \leq R \leq 1500$.
For $M \in \bar{2}_6$, 5,..89 are $\bar{2}_6$ primes and 7,..., 79 are $\bar{4}_6$primes.

R'^*	Row Numbers	Nos of Rows
0	500 520 550 560 590 670 710 800 850 880 920 940 980 1050 1060 1110 1130 1160 1260 1370	22
2	542 562 612 642 682 712 822 872 942 1022 1092 1132 1202 1222 1372 1382	16
8	528 578 588 628 688 758 798 828 978 1138 1158 1188 1218 1248 1258 1348 1398 1438	18
3	543 593 653 693 703 723 753 773 903 913 943 1033 1243 1293 1313 1423 1433 1473	18
5	555 565 655 675 705 775 835 975 1015 1045 1075 1095 1115 1145 1225 1265 1325 1335 1405 1495	20
7	577 597 637 667 707 737 747 787 837 907 917 957 1117 1127 1477	15

Table 7: Rows with twin primes for $500 \leq R \leq 1500$

We should emphasise that we are not using tests of primality or compositeness as such. We identify rows where either Class $\overline{2}_6$ or $\overline{4}_6$ or both have a composite. This is done by the simple linear equation (2.4).

Hans Riesel (1994) gives many methods for finding odd composite numbers. The class structure of these would then need to be established in order to identify the rows that they occupy within $\mathbb{Z}_6$, and obviously such rows would not contain twin primes (unless the composites were confined to Class $\overline{6}_6$). As well, those rows that have only one of the Class $\overline{2}_6$ or $\overline{4}_6$ sites occupied must contain a prime.

Here we use primes to build up composites from Equation (2.4) so that the factors are known and the characters of the rows within $\mathbb{Z}_6$ are detailed. This allows rows containing single primes to be identified easily. A knowledge of the $\mathbb{Z}_6$ integer/class/row structure should be useful more generally. For example, exceptions to Fermat compositeness tests for an integer n, or Carmichael numbers, can be given by (Riesel, 1994)

$$N = (6t+1)(12t+1)(18t+1)$$

with all factors being primes with a common t. All these factors belong to $\overline{4}_6$, as does N itself. The smallest Carmichael number is $561 = 3 \times 1 \times 17$, which is unique since $3|N \in \overline{6}_6$ always. Some questions follow:

Why do these numbers fall in $\overline{4}_6$?

Is there any pattern to the row occupancy?

Are there any primes in the rows occupied by the Carmichael numbers?

Do the REDs show any unique features (for a start $t^* \neq 2,3,4,7,8$)? and so on.

We should note finally that understanding the structure of integer arrays, how they are built up and fall into place within $\mathbb{Z}_6$ has been the primary aim here. The location of the primes and twin primes follow as a consequence. Thus the analysis given here will be of more general usefulness, than it would have otherwise been.

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