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EXPLICIT FORMULAE FOR THE n-TH TERM OF THE TWIN PRIME SEQUENCE

Mladen Vassilev - Missana
5, V. Hugo Str., Sofia–1124, Bulgaria

Abstract In this paper three different explicit formulae for the n-th term of the twin prime sequence are proposed and proved. The investigation continues [1].

Used denotations: $\mathcal{N}$ - the set of all natural numbers; $[x]$ - the greatest integer which is not greater than the real nonegative number x; ζ - Riemann's zeta function; Γ - the Euler's function gamma; $\pi_2(n)$ - the number of primes p such that $p \leq n$ and $p+2$ is also a prime; $p_2(n)$ - the n-th term of the twin prime sequence, i.e.,

$$p_2(1) = 3, p_2(2) = 5, p_2(3) = 7, p_2(4) = 11, p_2(5) = 13, p_2(6) = 17, p_2(7) = 19,$$

$$p_2(8) = 29, p_2(9) = 31, \ldots$$

We need the following result [1]:

Theorem 1: Let $n \geq 4$ be even. Then $p_2(n)$ has each one of the following three representations:

$$p_2(n) = 5 + \sum_{k=5}^{\infty}[\frac{1}{1 + H(k;n)}]; \tag{1}$$

$$p_2(n) = 5 - 2.\sum_{k=5}^{\infty}\zeta(-2.H(k;n)); \tag{2}$$

$$p_2(n) = 5 + \sum_{k=5}^{\infty}\frac{1}{\Gamma(1 - H(k;n))}, \tag{3}$$

where

$$H(k;n) = [\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}]. \tag{4}$$

In the present paper we shall prove the following

Theorem 2: Let $n \geq 4$ be integer. Then $p_2(n)$ has each one of the following three representations:

$$p_2(n) = 6 + (-1)^{n-1} + \sum_{k=5}^{\infty}[\frac{1}{1 + r(k;n)}]; \tag{1*}$$

$$p_2(n) = 6 + (-1)^{n-1} - 2.\sum_{k=5}^{\infty} \zeta(-2.r(k;n)); \tag{2*}$$

$$p_2(n) = 6 + (-1)^{n-1} + \sum_{k=5}^{\infty} \frac{1}{\Gamma(1 - r(k;n))}, \tag{3*}$$

where

$$r(k;n) = [\frac{\pi_2(k) - 1 + [\frac{n}{2}]}{2.[\frac{n}{2}]}]. \tag{4*}$$

Proof: Let $n \geq 4$ be even. Then $r(k;n) = H(k;n)$ and also $6 + (-1)^{n-1} = 5$. Therefore (1*) coincides with (1), (2*) coincides with (2), and (3*) coincides with (3), which proves the Theorem in this case.

Let $n > 4$ be odd. Then

$$r(k;n) = H(k;n-1). \tag{5}$$

Since $[\frac{n}{2}] = \frac{n-1}{2}$ and $2.[\frac{n}{2}] = n - 1$.

We have also tne relation

$$p_2(n) = 2 + p_2(n-1). \tag{6}$$

Since $p_2(n-1)$ and $p_2(n)$ are twin primes. But $n - 1$ is even and $n - 1 \geq 4$. Then we apply Theorem 1 with $n-1$ instead of n and from (5) and (6) the proof of Theorem 2 falls, because of the equality $6 + (-1)^{n-1} = 2 + 5$.

Finally, we observe that formulae (1*)-(3*) are explicit, because in [2] are proposed some different explicit formulae for $\pi_2(n)$ when $n \geq 5$. One of these formulae is given below:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{2(6k-2)! + (6k)!}{36k^2 - 1} - [\frac{2(6k-2)! + (6k)! + 2}{36k^2 - 1}]].$$

References:

[1] Vassilev–Missana, M. Three formulae for n-th prime and six for n-th term of twin primes. Notes on Number Theory and Discrete Mathematics, Vol. 7, 2001, No. 1, 15-20.

[2] Vassilev–Missana, M. Some new formulae for the twin primes counting function $\pi_2(n)$. Notes on Number Theory and Discrete Mathematics, Vol. 7, 2001, No. 1, 10-14.